

Fourier transform and the global Gan–Gross–Prasad conjecture for unitary groups

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Abstract

By the relative trace formula approach of Jacquet–Rallis, we prove the global Gan–Gross–Prasad conjecture for unitary groups under some local restrictions for the automorphic representations.

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Email: wzhang@math.columbia.edu. The author is partially supported by NSF Grant DMS #1001631, #1204365, #1301848, and a Sloan research fellowship.

[†]2000 *Mathematics Subject Classification*. Primary 11F67, 11F70, 22E55; Secondary 11G40, 22E50.

[‡]*Key words and phrases*. Automorphic period, Rankin–Selberg L-function, the global Gan–Gross–Prasad conjecture, Jacquet–Rallis relative trace formula, spherical character, orbital integrals, smooth transfer, uncertainty principle.

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1 Introduction to Main results

The studies of periods and heights related to automorphic forms and Shimura varieties have recently received a lot of attention. One pioneering example is the work of Harder–Langlands–Rapoport ([27]) on the Tate conjecture for Hilbert–Blumenthal modular surfaces. Another example which motivates the current paper is the Gross–Zagier formula. It concerns the study of the Neron–Tate heights of Heegner points or CM points: on the modular curve $X_0(N)$ by Gross and Zagier ([23]) in 1980s, on Shimura curves by S. Zhang in 1990’s, completed by Yuan–Zhang–Zhang ([60]) recently (also cf. Kudla–Rapoport–Yang ([41]), Bruinier–Ono ([6]) etc., in various perspectives). At almost the same time as the Gross–Zagier’s work, Waldspurger ([56]) discovered a formula that relates certain toric periods to the central value of L-functions on GL_2 , the same type L-function appeared in the Gross–Zagier formula. The Waldspurger formula and the Gross–Zagier formula are crucial in the study of the arithmetic of elliptic curves. In 1990’s, Gross and Prasad formulated a conjectural generalization of Waldspurger’s work to higher rank orthogonal groups ([21], [22]) (later refined by Ichino–Ikeda [31]). Recently, Gan, Gross and Prasad have generalized the conjectures further to classical groups ([14]) including unitary groups and symplectic groups. The conjectures are on the relation between period integrals and certain L-values. The main result of this paper is to confirm their conjecture for unitary groups under some local restrictions. A subsequent paper [63] is devoted to the refined conjecture for unitary groups.

In the following we describe the main results of the paper in more details.

Gan–Gross–Prasad conjecture for unitary groups. Let E/F be a quadratic extension of number fields with adèles denoted by $\mathbb{A} = \mathbb{A}_F$ and \mathbb{A}_E respectively. Let W be a (non-degenerate) Hermitian space of dimension n . We denote by $U(W)$ the corresponding unitary group, as an algebraic group over F . Let $G'_n = \mathrm{Res}_{E/F}\mathrm{GL}_n$ be the restriction of scalar of GL_n from E to F . Let v be a place of F and F_v the completion at v of F . Let π_v be an irreducible admissible representation of $U(W)(F_v)$. We recall the local base change map when a place v is split or the representation is unramified. If a place v of F is split in E/F , we may identify $G'_n(F_v)$ with $\mathrm{GL}_n(F_v) \times \mathrm{GL}_n(F_v)$ and identify $U(W)(F_v)$ with a subgroup consisting of elements of the form $(g, {}^t g^{-1})$, $g \in \mathrm{GL}_n(F_v)$, where ${}^t g$ is the transpose of g . Let p_1, p_2 be the two isomorphisms between $U(W)(F_v)$ with $\mathrm{GL}_n(F_v)$ induced by the two projections

from $\mathrm{GL}_n(F_v) \times \mathrm{GL}_n(F_v)$ to $\mathrm{GL}_n(F_v)$. We define the local base change $BC(\pi_v)$ to be the representation $p_1^* \pi_v \otimes p_2^* \pi_v$ of $G'(F_v)$ where $p_i^* \pi_v$ is a representation of $\mathrm{GL}_n(F_v)$ obtained by the isomorphism p_i . Note that when v is split, the local base change map is injective. When v is non-split and $U(W)$ is unramified at v , there is a local base change map at least when π_v is an unramified representation of $U(W)(F_v)$, cf. [14, §8]. Now let π be a cuspidal automorphic representation of $U(W)(\mathbb{A})$. An automorphic representation $\Pi = \otimes_v \Pi_v$ of $G'_n(\mathbb{A})$ is called the *weak base change* of π if Π_v is the local base change of π_v for all but finitely many places v where π_v is unramified ([25]). We will then denote it by $BC(\pi)$.

Throughout this article, we will assume the following hypothesis on the base change.

Hypothesis (*): *For all n , W and cuspidal automorphic π , the weak base change $BC(\pi)$ of π exists and satisfies the following local–global compatibility at all split places v : the v -component of $BC(\pi)$ is the local base change of π_v .*

Remark 1. This hypothesis should follow from the analogous work of Arthur on endoscopic classification for unitary groups. For quasi-split unitary groups, this has been recently carried out by Mok ([43]), whose appendix is relevant to our Hypothesis (*). A much earlier result of Harris–Labesse ([25, Theorem 2.2.2]) shows that the hypothesis is valid if (1) π have supercuspidal components at two split places, and (2) either n is odd or all archimedean places of F are complex.

Let W, W' be two Hermitian spaces of dimension n . Then for almost all v , the Hermitian spaces W_v and W'_v are isomorphic. We fix an isomorphism for almost every v , which induces an isomorphism between the unitary groups $U(W)(F_v)$ and $U(W')(F_v)$. We say that two automorphic representations π, π' of $U(W)(\mathbb{A})$ and $U(W')(\mathbb{A})$ respectively are *nearly equivalent* if $\pi_v \simeq \pi'_v$ for all but finitely many places v of F . Conjecturally, all automorphic representations in a *Vogan’s L -packet* ([14, §9, §10]) form precisely a single nearly equivalence class. By the strong multiplicity one theorem for GL_n , if π, π' are nearly equivalent, their weak base changes must be the same.

We recall the notion of (global) *distinction* following Jacquet. Let G be a reductive group over F and H a subgroup. Let $\mathcal{A}_0(G)$ be the space of cuspidal automorphic forms on $G(\mathbb{A})$. We define a period integral

$$\begin{aligned} \ell_H : \mathcal{A}_0(G) &\rightarrow \mathbb{C} \\ \phi &\mapsto \int_{(Z_G \cap H)(\mathbb{A})H(F)\backslash H(\mathbb{A})} \phi(h) dh \end{aligned}$$

whenever the integral makes sense. Here Z_G denotes the F -split torus of the center of G . Similarly, if χ is a character of $H(F)\backslash H(\mathbb{A})$, we define

$$\ell_{H,\chi}(\phi) = \int_{Z_G \cap H(\mathbb{A})H(F)\backslash H(\mathbb{A})} \phi(h) \chi(h) dh.$$

For a cuspidal automorphic representation π (viewed as a subrepresentation of $\mathcal{A}_0(G)$), we say that it is (χ -, resp.) *distinguished by H* if the linear functional ℓ_H ($\ell_{H,\chi}$, resp.) is nonzero when restricted to π . Even if the multiplicity one fails for G , this definition still makes sense as our π is understood as a pair (π, ι) where ι is an embedding of π into $\mathcal{A}_0(G)$.

To state the main result of this paper on the global Gan–Gross–Prasad conjecture ([14, §24]), we let (W, V) be a fixed pair of (non-degenerate) Hermitian spaces of dimension n and $n+1$ respectively, with an embedding $W \hookrightarrow V$. The embedding $W \hookrightarrow V$ induces an embedding of unitary groups $\iota : U(V) \hookrightarrow U(W)$. We denote by $\Delta_{U(W)}$ the image of $U(W)$ under the diagonal embedding into $U(V) \times U(W)$. Let π be a cuspidal automorphic representation of $U(V) \times U(W)$ with its weak base change Π . We define (cf. [14, §22])

$$(1.1) \quad L(s, \Pi, R) = L(s, \Pi_{n+1} \times \Pi_n),$$

where $L(s, \Pi_{n+1} \times \Pi_n)$ is the Rankin–Selberg L-function if we write $\Pi = \Pi_n \otimes \Pi_{n+1}$.

The main result of this paper is as follows, proved in §2.5 and §2.7.

Theorem 1.1. *Assume that Hypothesis (*) holds. Let π be a cuspidal automorphic representation of $U(V) \times U(W)$. Suppose that*

- (1) *Every archimedean place is split in E/F .*
- (2) *There exist two distinct places v_1, v_2 (non-archimedean) split in E/F such that π_{v_1}, π_{v_2} are supercuspidal.*

Then the following are equivalent

- (i) *The central L-value does not vanish: $L(1/2, \Pi, R) \neq 0$.*
- (ii) *There exists Hermitian spaces $W' \subset V'$ of dimension n and $n+1$ respectively, and an automorphic representation π' of $U(V') \times U(W')$ nearly equivalent to π , such that π' is distinguished by $\Delta_{U(W')}$.*

Remark 2. Note that we do not assume that the representation π' occurs with multiplicity one in the space of cuspidal automorphic forms $\mathcal{A}_0(U(V') \times U(W'))$ (though this is an expected property of the L-packet for unitary groups). By π' we do mean a subspace of $\mathcal{A}_0(U(V') \times U(W'))$.

The theorem confirms the *global* conjecture of Gan–Gross–Prasad ([14, §24]) for unitary group under the local restrictions (1) and (2). The two conditions are due to some technical issues we now briefly describe. Our approach is by a simple version of Jacquet–Rallis relative trace formulae (shortened as “RTF” in the rest of the paper). The first assumption is due to the fact that we only prove the existence of smooth transfer for a p -adic field (cf. Remark 3). The second assumption is due to the fact that we use a “cuspidal” test function at a split place and use a test function with nice support at another split place (cf. Remark 4). To remove the second assumption, one needs the fine spectral expansion of the RTF of Jacquet–Rallis, which seems to be a very difficult problem on its own. Towards this, there has been the recent work of Ichino and Yamana on the regularization of period integral [32].

Remark 3. In the archimedean case we have some partial result for the existence of smooth transfer (Theorem 3.14). If we assume the local–global compatibility of weak base change at a non-split archimedean place, we may replace the first assumption by the following: *if $v|\infty$ is non-split, then W, V are positive definite (hence π_v is finite dimensional) and*

$$\mathrm{Hom}_{U(W)(F_v)}(\pi_v, \mathbb{C}) \neq 0.$$

Remark 4. In Theorem 1.1, we may weaken the second condition to require only that π_{v_1} is supercuspidal and π_{v_2} is tempered.

Remark 5. We recall some by-no-means complete history related to this conjecture. In the lower rank cases, a lot of works have been done on the global Gan–Gross–Prasad conjecture for orthogonal groups: the work of Waldspurger on $\mathrm{SO}(2) \times \mathrm{SO}(3)$ ([56]), the work of Garrett ([16]), Piatetski-Shapiro–Rallis, Garret–Harris, Harris–Kudla ([24]), Gross–Kudla ([20]), and Ichino ([30]) on the case of $\mathrm{SO}(3) \times \mathrm{SO}(4)$ or the so-called *Jacquet’s conjecture*, the work of Gan–Ichino on some cases of $\mathrm{SO}(4) \times \mathrm{SO}(5)$ ([13]). For the case of higher rank, Ginzburg–Jiang–Rallis ([18], [19] etc.) prove one direction of the conjecture for some representations in both the orthogonal and the unitary cases.

Remark 6. The original local Gross–Prasad conjecture ([21],[22], for the orthogonal case) for p -adic fields has also been resolved in a series of papers by Waldspurger and Mœglin ([59], [44] etc.). It is extended to the unitary case ([14]) by Beuzart-Plessis ([7], [8]). But in our paper we will not need this. According to this local conjecture of Gan–Gross–Prasad for unitary groups and the expected multiplicity-one property of π in the cuspidal spectrum, such *relevant* ([14]) pair (W', V') and π' in Theorem 1.1 should be unique (if it exists).

Remark 7. Ichino and Ikeda stated a refinement of the Gross–Prasad conjecture in [31] for the orthogonal case. N. Harris ([26]) extended the refinement to the unitary case of the Gan–Gross–Prasad conjecture. The approach of trace formula and the major local ingredients in this paper will be used in a subsequent paper ([63]) to establish the refinement of the Gan–Gross–Prasad conjecture for unitary groups under certain local conditions.

An application to non-vanishing of central L-values. We have an application to the existence of non-vanishing twist of Rankin–Selberg L-function. It may be of independent interest.

Theorem 1.2. *Let E/F be a quadratic extension of number fields such that all archimedean places are split. Let σ be a cuspidal automorphic representation of $\mathrm{GL}_{n+1}(\mathbb{A}_E)$, $n \geq 1$. Assume that σ is a weak base change of an automorphic representation π of some unitary group $U(V)$ where π_v is locally supercuspidal at two split places v of F . Then there exists a cuspidal automorphic representation τ of $\mathrm{GL}_n(\mathbb{A}_E)$ such that the central value of the Rankin–Selberg L-function does not vanish:*

$$L\left(\frac{1}{2}, \sigma \times \tau\right) \neq 0.$$

This is proved in §2.8.

Flicker–Rallis conjecture. Let $\eta = \eta_{E/F}$ be the quadratic character of $F^\times \backslash \mathbb{A}^\times$ associated to the quadratic extension E/F by class field theory. By abuse of notation, we will denote by η the quadratic character $\eta \circ \det$ (\det being the determinant map) of $\mathrm{GL}_n(\mathbb{A})$.

Conjecture 1.3 (Flicker–Rallis, [11]). *An automorphic cuspidal representation Π on $\mathrm{GL}_n(\mathbb{A}_E)$ is a weak base change from a cuspidal automorphic π on some unitary group in n -variables if and only if it is distinguished ($\eta_{E/F}$ -distinguished, resp.) by $\mathrm{GL}_{n,F}$ if n is odd (even, resp.).*

Another result of the paper is to confirm one direction of Flicker-Rallis conjecture under the same local restrictions as in Theorem 1.1. In fact, this result is used in the proof of Theorem 1.1.

Theorem 1.4. *Let π be a cuspidal automorphic representation of $U(W)(\mathbb{A})$ satisfying:*

- (1) *Every archimedean place is split in E/F .*
- (2) *There exist two distinct places v_1, v_2 (non-archimedean) split in E/F such that π_{v_1}, π_{v_2} are supercuspidal.*

Then the weak base change $BC(\pi)$ is (η -, resp.) distinguished by $GL_{n,F}$ if n is odd (even, resp.).

This is proved in §2.6.

Remark 8. If Π is distinguished by $GL_{n,F}$, then Π is conjugate self-dual ([11]). Moreover, the partial Asai L-function has a pole at $s = 1$ if and only if Π is distinguished by $GL_{n,F}$ ([10], [12]). In [17], it is further proved that if the central character of Π is distinguished, then Π is conjugate self-dual if and only if Π is distinguished (resp., η -distinguished) if n is odd (resp., even). ¹

We briefly describe the contents of each section. In section 2, we prove the main theorems assuming the existence of smooth transfer. In section 3 we reduce the existence of smooth transfer on groups to the same question on “Lie algebras” (an infinitesimal version). In section 4, we show the existence of smooth transfer on Lie algebras for a p -adic field .

Acknowledgements. The author would like to thank A. Aizenbud, D. Goldfeld, B. Gross, A. Ichino, H. Jacquet, D. Jiang, E. Lapid, Y. Liu, D. Prasad, Y. Sakellaridis, B. Sun, Y. Tian, A. Venkatesh, H. Xue, X. Yuan, Z. Yun, S. Zhang, X. Zhu for their help during the preparation of the paper. He also thanks the Morningside Center of Mathematics of Chinese Academy of Sciences, the Mathematical Science Center of Tsinghua University for their hospitality and support where some part of the paper was written. Finally, the author thanks the anonymous referee for several useful comments.

Notations and conventions. We list some notations and convention used throughout this paper. The others will be introduced as we meet them.

Let F be a number field or a local field, and let E be a semisimple quadratic F -algebra, and moreover, a field if F is a number field.

For a smooth variety X over a local field F we endow $X(F)$ with the analytic topology. We denote by $\mathcal{C}_c^\infty(X(F))$ the space of smooth (locally constant if F is non-archimedean) functions with compact support.

Some groups are as follows:

¹As Lapid points out to the author, the work of Ginzburg–Rallis–Soudry on automorphic descent already shows that for a cuspidal Π of $GL_n(\mathbb{A}_E)$, its Asai L-function has a pole at $s = 1$ if and only if Π is the base change from some π on a unitary group. In addition, the work of Arthur, extended to unitary groups, should also prove this. But our proof of Theorem 1.4 is different from theirs and may be of independent interest.

- *The general linear case.* We will consider the F -algebraic group

$$(1.2) \quad G' = \text{Res}_{E/F}(\text{GL}_{n+1} \times \text{GL}_n)$$

and two subgroups: H'_1 is the diagonal embedding of $\text{Res}_{E/F}\text{GL}_n$ (where GL_n is embedded into GL_{n+1} by $g \mapsto \text{diag}[g, 1]$) and $H'_2 = \text{GL}_{n+1, F} \times \text{GL}_{n, F}$ embedded into G' in the obvious way. In this paper for an F -algebraic group H , we will denote by Z_H the center of H . We note that $Z_{G'} \cap Z_{H'_1}$ is trivial.

- *The unitary case.* We will consider a pair of Hermitian spaces over the quadratic extension E of F : V and a codimension one subspace W . Suppose that W is of dimension n . Without loss of generality, we may and do always assume

$$(1.3) \quad V = W \oplus Eu,$$

where u has norm one: $(u, u) = 1$. In particular, the isometric class of V is determined by W . We have an obvious embedding of unitary groups $U(W) \hookrightarrow U(V)$. Let

$$(1.4) \quad G = G^W = U(V) \times U(W)$$

and let $\Delta : U(W) \hookrightarrow G$ be the diagonal embedding. Denote by $H = \Delta_{U(W)}$ (or H_W to emphasize the dependence on W) the image of Δ , as a subgroup of G .

For a number field F , let

$$(1.5) \quad \eta = \eta_{E/F} : F^\times \backslash \mathbb{A}^\times \rightarrow \{\pm 1\}$$

be the quadratic character associated to E/F by class field theory. By abuse of notation we will also denote by η the character of $H'_2(\mathbb{A})$ defined by $\eta(h) := \eta(\det(h_1))$ ($\eta(\det(h_2))$), resp.) if $h = (h_1, h_2) \in \text{GL}_{n+1}(\mathbb{A}) \times \text{GL}_n(\mathbb{A})$ and n is odd (even, resp.). Fix a character

$$(1.6) \quad \eta' : E^\times \backslash \mathbb{A}_E^\times \rightarrow \mathbb{C}^\times$$

(not necessarily quadratic) such that its restriction

$$\eta'|_{\mathbb{A}^\times} = \eta.$$

We similarly define the local analogue η_v, η'_v .

Let F be a field of character zero. For a reductive group H acting on an affine variety X , we say that a point $x \in X(F)$ is:

- *H -semisimple* if Hx is Zariski closed in X (when F is a local field, equivalently, $H(F)x$ is closed in $X(F)$ for the analytic topology, cf. [2, Theorem 2.3.8]).
- *H -regular* if the stabilizer H_x of x has the minimal dimension.

If no confusion, we will simply use the words “semisimple” and “regular”. We say that x is *regular semisimple* if it is regular and semisimple. In this paper, we will be interested in the following two cases

- $X = G$ is a reductive group and $H = H_1 \times H_2$ is a product of two reductive subgroups of G where H_1 (H_2 , resp.) acts by left (right, resp.) multiplication.
- $X = V$ is a vector space (considered as an affine variety) with an action by a reductive group H .

For $h \in H$ and $x \in X$, we will usually write (especially in an orbital integral)

$$(1.7) \quad x^h = h \cdot x$$

for the h -translation of x .

For later use, we also recall that the *categorical quotient* of X by H (cf. [2], [45]) consists of a pair (Y, π) where Y is an algebraic variety over F and $\pi : X \rightarrow Y$ is an H -morphism with the following universal property: for any pair (Y', π') with an H -morphism $\pi' : X \rightarrow Y'$, there exists a unique morphism $\phi : Y \rightarrow Y'$ such that $\pi' = \phi \circ \pi$. If such a pair exists, then it is unique up to a canonical isomorphism. When X is affine (in all our cases), the categorical quotient always exists. Indeed we may construct as follows. Consider the affine variety

$$X//H := \text{Spec } \mathcal{O}(X)^H$$

together with the obvious quotient morphism

$$\pi = \pi_{X,H} : X \rightarrow \text{Spec } \mathcal{O}(X)^H.$$

Then $(X//H, \pi)$ is a categorical quotient of X by H . By abuse of notation, we will also let π denote the induced map $X(F) \rightarrow (X//H)(F)$ if no confusion arises.

Below we list some other notations.

- M_n : $n \times n$ -matrices.
- F_n (F^n , resp.): the n -dimensional F -vector space of row (column, resp.) vectors.
- $e = e_{n+1} = (0, \dots, 0, 1) \in F_{n+1}$ is a $1 \times (n+1)$ -row vector and $e^* \in F^{n+1}$ its transpose.
- For a p -adic local field F , we denote by $\varpi = \varpi_F$ a fixed uniformizer.
- For E/F be a (separable) finite extension, we denote by $\text{tr} = \text{tr}_{E/F} : E \rightarrow F$ the trace map and $N = N_{E/F} : E^\times \rightarrow F^\times$ the norm map. Let E^1 (NE^\times , resp.) be the kernel (the image, resp.) of the norm map.

2 Relative trace formulae of Jacquet–Rallis

2.1 Orbital integrals

We first introduce the local orbital integrals appearing in the relative trace formulae of Jacquet–Rallis. We refer to [62, sec. 2] on important properties of orbits (namely, double cosets). Later on in §3 we will also recall some of them. We now let F be a local field of characteristic zero. And let E be a quadratic semisimple F -algebra, i.e., E is either a quadratic field extension of F or $E \simeq F \times F$.

The general linear case. We start with the *general linear case*. If an element $\gamma \in G'(F)$ is $H'_1 \times H'_2$ -regular semisimple, for simplicity we will say that it is regular semisimple. For a regular semisimple $\gamma \in G'(F)$ and a test function $f' \in \mathcal{C}_c^\infty(G'(F))$, we define its orbital integral as:

$$(2.1) \quad O(\gamma, f') := \int_{H'_1(F)} \int_{H'_2(F)} f'(h_1^{-1}\gamma h_2) \eta(h_2) dh_1 dh_2.$$

This depends on the choice of Haar measure. But in this paper, the choice of measure is not crucial since we will only concern non-vanishing problem. In the following, we always pre-assume that we have made a choice of a Haar measure on each group.

The integral (2.1) is absolutely convergent, and η -twisted invariant in the following sense

$$(2.2) \quad O(h_1^{-1}\gamma h_2, f') = \eta(h_2)O(\gamma, f'), \quad h_1 \in H'_1(\mathbb{A}), h_2 \in H'_2(\mathbb{A}).$$

We may simplify the orbital integral as follows. Identify $H'_1 \backslash G'$ with $\text{Res}_{E/F} \text{GL}_{n+1}$. Let S_{n+1} be the subvariety of $\text{Res}_{E/F} \text{GL}_{n+1}$ defined by the equation $s\bar{s} = 1$ where \bar{s} denotes the entry-wise Galois conjugation of $s \in \text{Res}_{E/F} \text{GL}_{n+1}$. By Hilbert Satz-90, we have an isomorphism of two affine varieties

$$\text{Res}_{E/F} \text{GL}_{n+1} / \text{GL}_{n+1, F} \simeq S_{n+1},$$

induced by the following morphism ν between two F -varieties:

$$(2.3) \quad \nu : \text{Res}_{E/F} \text{GL}_{n+1} \rightarrow S_{n+1}$$

$$(2.4) \quad g \mapsto g\bar{g}^{-1}.$$

Moreover, we have a homeomorphism on the level of F -points:

$$(2.5) \quad \text{GL}_{n+1}(E) / \text{GL}_{n+1}(F) \simeq S_{n+1}(F).$$

We may integrate f' over $H'_1(F)$ to get a function on $\text{Res}_{E/F} \text{GL}_{n+1}(F)$:

$$\bar{f}'(x) := \int_{H'_1(F)} f'(h_1(x, 1)) dh_1, \quad x \in \text{Res}_{E/F} \text{GL}_{n+1}(F).$$

We first assume that n is odd. Then the character η on H'_2 is indeed only nontrivial on the component $\text{GL}_{n+1, F}$. We may introduce a function \tilde{f}' on $S_{n+1}(F)$ as follows: when $\nu(x) = s \in S_{n+1}(F)$, we define

$$\tilde{f}'(s) := \int_{\text{GL}_{n+1}(F)} \bar{f}'(xg) \eta'(xg) dg.$$

Then $\tilde{f}' \in \mathcal{C}_c^\infty(S_{n+1}(F))$ and all functions in $\mathcal{C}_c^\infty(S_{n+1}(F))$ arise this way. Now it is easy to see that for $\gamma = (\gamma_1, \gamma_2) \in G'(F) = \text{GL}_{n+1}(E) \times \text{GL}_n(E)$:

$$(2.6) \quad O(\gamma, f') = \eta'(\det(\gamma_1 \gamma_2^{-1})) \int_{\text{GL}_n(F)} \tilde{f}'(h^{-1}sh) \eta(h) dh, \quad s = \nu(\gamma_1 \gamma_2^{-1}).$$

If n is even, we simply define in the above

$$\tilde{f}'(s) := \int_{\mathrm{GL}_{n+1}(F)} \bar{f}'(xg)dg, \quad \nu(x) = s.$$

We then have for $\gamma = (\gamma_1, \gamma_2)$

$$(2.7) \quad O(\gamma, f') = \int_{\mathrm{GL}_n(F)} \tilde{f}'_v(h^{-1}sh)\eta(h)dh, \quad s = \nu(\gamma_1\gamma_2^{-1}).$$

An element $\gamma = (\gamma_1, \gamma_2) \in G'(F)$ is $H'_1 \times H'_2$ -regular semisimple if and only if $s = \nu(\gamma_1\gamma_2^{-1}) \in S_{n+1}(F)$ is $\mathrm{GL}_{n,F}$ -regular semisimple. We also recall that, by [51, §6], an element $s \in S_{n+1}(F)$ is $\mathrm{GL}_{n,F}$ -regular semisimple if and only if the following discriminant does not vanish

$$(2.8) \quad \Delta(s) := \det(es^{i+j}e^*)_{i,j=0,1,\dots,n} \neq 0,$$

where $e = (0, \dots, 0, 1)$ is a row vector and e^* its transpose.

To deal with the center of G' , we will also need to consider the action of $\mathcal{H} := Z_{G'}H'_1 \times H'_2$ on G' . Though the categorical quotient of G' by $Z_{G'}H'_1 \times H'_2$ exists, we are not sure how to explicitly write down a set of generators of invariant regular functions nor how to determine when γ is $Z_{G'}H'_1 \times H'_2$ -regular semisimple. But we may give an explicit Zariski open subset consisting of $Z_{G'}H'_1 \times H'_2$ -regular semisimple elements. It suffices to work with the space S_{n+1} . Then we have the induced action of $Z_{G'} \times \mathrm{GL}_{n,F}$ on S_{n+1} :

- $h \in \mathrm{GL}_{n,F}$ acts by the conjugation,
- $z = (z_1, z_2) \in Z_{G'} \simeq (E^\times)^2$ acts by Galois-conjugate conjugation by $z_2^{-1}z_1$:

$$z \circ s = (z_2^{-1}z_1)s(\overline{z_1^{-1}z_2}).$$

The two subgroups $Z_{\mathrm{GL}_{n+1,F}} \subset Z_{G'}$ and $\{((1, z_2), z_2) \in Z_{G'} \times \mathrm{GL}_{n,F} \mid z_2 \in Z_{\mathrm{GL}_{n,F}}\}$ clearly act trivially on S_{n+1} . We let Z_0 denote their product. We may write

$$s = \begin{pmatrix} A & b \\ c & d \end{pmatrix} \in S_{n+1}(F),$$

where $A \in M_n(E)$, $b \in M_{n,1}(E)$, $c \in M_{1,n}(E)$, $d \in E$. Then we have the following $Z_{G'} \times \mathrm{GL}_{n,F}$ -invariant polynomials on S_{n+1} :

$$(2.9) \quad N_{E/F}(\mathrm{tr}(A)), \quad N_{E/F}d.$$

We say that s is *Z-regular semisimple* if s is GL_n -regular semisimple and the above two invariants are invertible in E . When E is a field, this is equivalent to:

$$\mathrm{tr}(A) \neq 0, \quad d \neq 0, \quad \Delta(s) \neq 0.$$

Otherwise we understand “ $\neq 0$ ” as “ $\in E^\times$ ” in these inequalities. The *Z-regular semisimple locus*, denoted by \mathcal{Z} , clearly forms a Zariski open dense subset in S_{n+1} .

Lemma 2.1. *If s is Z -regular semisimple, the stabilizer of s is precisely Z_0 and its $Z_{G'} \times \mathrm{GL}_{n,F}$ -orbit is closed. In particular, a Z -regular semisimple element is $Z_{G'} \times \mathrm{GL}_{n,F}$ -regular semisimple.*

Proof. Suppose that $(z, h) \circ s = s$. As $\mathrm{tr}(A) \neq 0, d \neq 0$, up to modification by elements in Z_0 , we may assume that $z = 1$. Then the first assertion follows from the fact that the stabilizer of s is trivial for the GL_n -action on S_{n+1} when $\Delta(s) \neq 0$. When $\mathrm{tr}(A) \neq 0, d \neq 0$, besides $N_{E/F}(\mathrm{tr}(A)), N_{E/F}d$, the following are also $Z_{G'} \times \mathrm{GL}_{n,F}$ -invariant:

$$\frac{\mathrm{tr} \wedge^i A}{(\mathrm{tr}(A))^i}, \quad \frac{cA^j b}{(\mathrm{tr}(A))^{j+1}d}, \quad 1 < i \leq n, 0 \leq j \leq n-1.$$

Then we *claim* that two Z -regular semisimple s, s' are in the same $Z_{G'} \times \mathrm{GL}_{n,F}$ -orbit if and only if they have the same invariants (listed above). One direction is obvious. For the other direction, we now assume that s, s' are Z -regular semisimple and have the same invariants. In particular, the values of $N_{E/F}(\mathrm{tr}(A)), N_{E/F}d$ are the same. Replacing s' by zs' for a suitable $z \in Z_{G'}$, we may assume that s' and s have the same $\mathrm{tr}(A)$ and d . Then s, s' have the same values of $\mathrm{tr} \wedge^i A, 1 \leq i \leq n$ and $cA^j b, 0 \leq j \leq n-1$. Then by [62, §2], s and s' are conjugate by $\mathrm{GL}_{n,F}$ since they are also $\mathrm{GL}_{n,F}$ -regular semisimple. This proves the claim. Therefore, the $Z_{G'} \times \mathrm{GL}_{n,F}$ -orbit of s consists of $s \in S_{n+1}$ such that for a fixed tuple $(\alpha, \beta, \alpha_i, \beta_j)$

$$N_{E/F}(\mathrm{tr}(A)) = \alpha, N_{E/F}d = \beta,$$

and

$$\alpha_i = \frac{\mathrm{tr} \wedge^i A}{(\mathrm{tr}(A))^i}, \beta_j = \frac{cA^j b}{(\mathrm{tr}(A))^{j+1}d}, 1 < i \leq n, 0 \leq j \leq n-1.$$

The second set of conditions can be rewritten as

$$\mathrm{tr} \wedge^i A - \alpha_i (\mathrm{tr}(A))^i = 0, cA^j b - \beta_j (\mathrm{tr}(A))^{j+1}d = 0,$$

for $1 < i \leq n, 0 \leq j \leq n-1$. This shows that the $Z_{G'} \times \mathrm{GL}_{n,F}$ -orbit of s is Zariski closed. \square

Let χ' be a character of the center $Z_{G'}(F)$ that is trivial on $Z_{H_2'}(F)$. If an element $\gamma \in G'(F)$ is Z -regular semisimple, we define the χ' -orbital integral of $f' \in \mathcal{C}_c^\infty(G'(F))$ as:

$$(2.10) \quad O_{\chi'}(\gamma, f') := \int_{H_1'(F)} \int_{Z_{H_2'}(F) \backslash H_2'(F)} \int_{Z_{G'}(F)} f'(h_1^{-1} z^{-1} \gamma h_2) \chi'(z) \eta(h_2) dz dh_1 dh_2.$$

The integral is absolutely convergent.

The unitary case. We now consider the *unitary case*. Similarly, we will simply use the term “regular semisimple” relative to the action of $H \times H$ on $G = U(V) \times U(W)$. For a regular semisimple $\delta \in G(F)$ and $f \in \mathcal{C}_c^\infty(G(F))$, we define its orbital integral

$$(2.11) \quad O(\delta, f) = \int_{H(F) \times H(F)} f(x^{-1} \delta y) dx dy.$$

The integral is absolutely convergent. Similar to the general linear case, we may simplify the orbital integral $O(\delta, f)$. We introduce a new function on $U(V)(F)$:

$$(2.12) \quad \bar{f}(g) = \int_{U(W)(F)} f((g, 1)h)dh, \quad g \in U(V)(F).$$

Then for $\delta = (\delta_{n+1}, \delta_n) \in G(F)$, we may rewrite (2.11) as

$$(2.13) \quad O(\delta, f) = \int_{U(W)(F)} \bar{f}(y^{-1}(\delta_{n+1}\delta_n^{-1})y)dy.$$

We thus have the action of $U(W)$ on $U(V)$ by conjugation. An element $\delta = (\delta_{n+1}, \delta_n) \in G(F)$ is $H \times H$ -regular semisimple if and only if $\delta_{n+1}\delta_n^{-1} \in U(V)(F)$ is $U(W)$ -regular semisimple for the conjugation action. We recall that, by [62, §2], an element $\delta \in U(V)(F)$ is $U(W)$ -regular semisimple if and only if the vectors $\delta^i u \in V$, $i = 0, 1, \dots, n$, form an E -basis of V , where u is any non-zero vector in the line $W^\perp \subset V$ (cf. (1.3)). To deal with the center, we also need to consider the action of the center Z_G . Similar to the general linear case, we define the notion of Z -regular semisimple in terms the invariants in (2.9) where we view $\delta \in U(V)$ as an element in $\text{GL}(V)$. Then Lemma 2.1 easily extends to the unitary case. Let χ be a character of the center $Z_G(F)$. If an element $\delta \in G(F)$ is Z -regular semisimple, we define the χ -orbital integral as:

$$(2.14) \quad O_\chi(\delta, f) := \int_{H(F) \times H(F)} \int_{Z_G(F)} f(x^{-1}z\delta y)\chi(z) dz dx dy.$$

The integral is absolutely convergent.

2.2 RTF on the general linear group

Now we recall the construction of Jacquet–Rallis’ RTF on the general linear side ([39]). Let E/F be a quadratic extension of *number fields*. Fix a Haar measure on $Z_{G'}(\mathbb{A})$, $H'_i(\mathbb{A})$ ($i = 1, 2$) etc. and the counting measure on $Z_{G'}(F)$, $H'_i(F)$ ($i = 1, 2$) etc..

For $f' \in \mathcal{C}_c^\infty(G'(\mathbb{A}))$, we define a kernel function

$$K_{f'}(x, y) = \sum_{\gamma \in G'(F)} f'(x^{-1}\gamma y).$$

For a character χ' of $Z_{G'}(F) \backslash Z_{G'}(\mathbb{A})$, we define the χ' -part of the kernel function

$$K_{f', \chi'}(x, y) = \int_{Z_{G'}(F) \backslash Z_{G'}(\mathbb{A})} \sum_{\gamma \in G'(F)} f'(x^{-1}z^{-1}\gamma y)\chi'(z)dz.$$

We then consider a distribution on $G'(\mathbb{A})$:

$$(2.15) \quad I(f') = \int_{H'_1(F) \backslash H'_1(\mathbb{A})} \int_{H'_2(F) \backslash H'_2(\mathbb{A})} K_{f'}(h_1, h_2)\eta(h_2)dh_1dh_2.$$

Similarly, for a character χ' of $Z_{G'}(F)\backslash Z_{G'}(\mathbb{A})$ that is trivial on $Z_{H'_2}(\mathbb{A})$, we define the χ' -part of the distribution

$$(2.16) \quad I_{\chi'}(f') = \int_{H'_1(F)\backslash H'_1(\mathbb{A})} \int_{Z_{H'_2}(\mathbb{A})H'_2(F)\backslash H'_2(\mathbb{A})} K_{f',\chi}(h_1, h_2)\eta(h_2)dh_1dh_2.$$

For the convergence of the integral, we will consider a subset of test functions f' . We say that a function $f' \in \mathcal{C}_c^\infty(G'(\mathbb{A}))$ is *nice* with respect to χ' if it is decomposable $f' = \otimes_v f'_v$ and satisfies:

- For at least one place v_1 , the test function $f'_{v_1} \in \mathcal{C}_c^\infty(G'(F_{v_1}))$ is *essentially a matrix coefficient* of a supercuspidal representation with respect to χ'_{v_1} . This means that the function on $G(F_{v_1})$

$$f'_{v_1, \chi'_{v_1}}(g) := \int_{Z_G(F_{v_1})} f'_{v_1}(gz)\chi'_{v_1}(z)dz$$

is a matrix coefficient of a supercuspidal representation of $G(F_{v_1})$. In particular, we require that v_1 is non-archimedean.

- For at least one place $v_2 \neq v_1$, the test function f'_{v_2} is supported on the locus of Z -regular semisimple elements of $G'(F_{v_2})$. The place v_2 is not required to be non-archimedean.

Lemma 2.2. *Let χ' be a (unitary) character of $Z_{G'}(F)\backslash Z_{G'}(\mathbb{A})$ that is trivial on $Z_{H'_2}(\mathbb{A})$. Suppose that $f' = \otimes_v f'_v$ is nice with respect to χ' .*

- *As a function on $H'_1(\mathbb{A}) \times H'_2(\mathbb{A})$, $K_{f'}(h_1, h_2)$ is compactly supported modulo $H'_1(F) \times H'_2(F)$. In particular, the integral $I(f')$ converges absolutely.*
- *As a function on $H'_1(\mathbb{A}) \times H'_2(\mathbb{A})$, $K_{f', \chi'}(h_1, h_2)$ is compactly supported modulo $H'_1(F) \times H'_2(F)Z_{H'_2}(\mathbb{A})$. In particular, the integral $I_{\chi'}(f')$ converges absolutely.*

Proof. The kernel function $K_{f'}$ can be written as

$$\sum_{\gamma \in H'_1(F)\backslash G'(F)/H'_2(F)} \sum_{(\gamma_1, \gamma_2) \in H'_1(F) \times H'_2(F)} f'(h_1^{-1}\gamma_1^{-1}\gamma\gamma_2h_2),$$

where the outer sum is over a complete set of representatives γ of regular semisimple $H'_1(F) \times H'_2(F)$ -orbits. First we claim that in outer sum only finite many terms have non-zero contribution. Let $\Omega \subset G'(\mathbb{A})$ be the support of f' . Note that the invariants of $G'(\mathbb{A})$ defines a continuous map from $G'(\mathbb{A})$ to $X(\mathbb{A})$ where X is the categorical quotient of G' by $H'_1 \times H'_2$. So the image of the compact set Ω will be a compact set in $X(\mathbb{A})$. On the other hand the image of $h_1^{-1}\gamma_1^{-1}\gamma\gamma_2h_2$ is in the discrete set $X(F)$. Moreover for a fixed $x \in X(F)$ there is at most one $H'_1(F) \times H'_2(F)$ double coset with given invariants. This shows the outer sum has only finite many non-zero terms.

It remains to show that for a fixed $\gamma_0 \in G'(F)$, the function on $H'_1(\mathbb{A}) \times H'_2(\mathbb{A})$ defined by $(h_1, h_2) \mapsto f'(h_1^{-1}\gamma_0h_2)$ has compact support. Consider the continuous map $H'_1(\mathbb{A}) \times H'_2(\mathbb{A}) \rightarrow G'(\mathbb{A})$ given by $(h_1, h_2) \mapsto h_1^{-1}\gamma_0h_2$. When γ is regular semisimple, this defines an homeomorphism onto a closed subset of $G'(\mathbb{A})$. This implies the desired compactness and completes the proof the first assertion. The second one is similarly proved using the Z -regular semi-simplicity. \square

The last lemma allows us to decompose the distribution $I(f')$ in (2.15) into a finite sum of orbital integrals

$$I(f') = \sum_{\gamma} O(\gamma, f'),$$

where the sum is over regular semisimple $\gamma \in H'_1(F) \backslash G'(F) / H'_2(F)$ and

$$(2.17) \quad O(\gamma, f') := \int_{H'_1(\mathbb{A})} \int_{H'_2(\mathbb{A})} f'(h_1^{-1} \gamma h_2) \eta(h_2) dh_1 dh_2.$$

If $f' = \otimes_v f'_v$ is decomposable, we may decompose the orbital integral as a product of local orbital integrals:

$$O(\gamma, f') = \prod_v O(\gamma, f'_v),$$

where $O(\gamma, f'_v)$ is defined in (2.1). Similarly, we have a decomposition for the χ' -part $I_{\chi'}(f')$ in (2.16)

$$I_{\chi'}(f') = \sum_{\gamma} O_{\chi'}(\gamma, f'),$$

where the sum is over regular semisimple $\gamma \in Z_{G'}(F) H'_1(F) \backslash G'(F) / H'_2(F)$.

For a cuspidal automorphic representation Π of $G'(\mathbb{A})$ whose central character is trivial on $Z_{H'_2}(\mathbb{A})$, we define a (global) *spherical character*

$$(2.18) \quad I_{\Pi}(f') = \sum_{\phi \in \mathcal{B}(\Pi)} \left(\int_{H'_1(F) \backslash H'_1(\mathbb{A})} \Pi(f') \phi(x) dx \right) \left(\int_{Z_{H'_2}(\mathbb{A}) H'_2(F) \backslash H'_2(\mathbb{A})} \overline{\phi(x)} dx \right),$$

where the sum is over an orthonormal basis $\mathcal{B}(\Pi)$ of Π .

We are now ready to state a simple RTF for nice test functions on $G'(\mathbb{A})$:

Theorem 2.3. *Let χ' be a (unitary) character of $Z_{G'}(F) \backslash Z_{G'}(\mathbb{A})$ that is trivial on $Z_{H'_2}(\mathbb{A})$. If $f' \in \mathcal{C}_c^{\infty}(G'(\mathbb{A}))$ is nice with respect to χ' , then we have an equality*

$$\sum_{\gamma} O_{\chi'}(f') = \sum_{\Pi} I_{\Pi}(f'),$$

where the sum on the left hand side runs over all Z -regular semisimple

$$\gamma \in H'_1(F) \backslash G'(F) / Z_G(F) H'_2(F)$$

and the sum on the right hand side runs over all cuspidal automorphic representations Π of $G'(\mathbb{A})$ with central character χ' .

Proof. It suffices to treat the spectral side. Let ρ be the right translation of $G'(\mathbb{A})$ on $L^2(G', \chi')$ (cf. [50] for this notation). Since f'_{v_1} is essentially a matrix coefficient of a super-cuspidal representation, by [50, Proposition 1.1], $\rho(f)$ acts by zero on the orthogonal complement of

the cuspidal part $L_0^2(G', \chi')$. We obtain that the kernel function is an absolute convergent sum

$$(2.19) \quad K_{f', \chi'}(x, y) = \sum_{\phi} \rho(f') \phi(x) \overline{\phi(y)},$$

where the sum runs over an orthonormal basis of the cuspidal part $L_0^2(G', \chi')$. We may further assume that the ϕ 's are all in $\mathcal{A}_0(G', \chi')$. This yields an absolutely convergent sum

$$I_{\chi'}(f') = \sum_{\Pi} I_{\Pi}(f'),$$

where Π runs over automorphic cuspidal representations of $G'(\mathbb{A})$ with central character χ' . \square

2.3 RTF on unitary groups

We now recall the RTF of Jacquet–Rallis in the unitary case. For $f \in \mathcal{C}_c^\infty(G(\mathbb{A}))$ we consider a kernel function

$$K_f(x, y) = \sum_{\gamma \in G(F)} f(x^{-1}\gamma y),$$

and a distribution

$$J(f) := \int_{H(F) \backslash H(\mathbb{A})} \int_{H(F) \backslash H(\mathbb{A})} K_f(x, y) dx dy.$$

Fix a (necessarily unitary) character $\chi = (\chi_{n+1}, \chi_n) : Z_G(F) \backslash Z_G(\mathbb{A}) \rightarrow \mathbb{C}^\times$. We introduce the χ -part of the kernel function

$$K_{f, \chi}(x, y) = \int_{Z_G(F) \backslash Z_G(\mathbb{A})} K_f(zx, y) \chi(z) dz = \int_{Z_G(F) \backslash Z_G(\mathbb{A})} K_f(x, z^{-1}y) \chi(z) dz,$$

and a distribution

$$J_\chi(f) := \int_{H(F) \backslash H(\mathbb{A})} \int_{H(F) \backslash H(\mathbb{A})} K_{f, \chi}(x, y) dx dy.$$

Note that the center Z_G is an anisotropic torus and its intersection with H is trivial.

Similar to the general linear case, we will consider a simple RTF for a subset of test functions $f \in \mathcal{C}_c^\infty(G(\mathbb{A}))$. We say that a function $f \in \mathcal{C}_c^\infty(G(\mathbb{A}))$ is *nice* with respect to χ if $f = \otimes_v f_v$ satisfies

- For at least one place v_1 , the test function f_{v_1} is essentially a matrix coefficient of a supercuspidal representation with respect to χ_{v_1} . This means that the function

$$f_{v_1, \chi_{v_1}}(g) = \int_{Z_G(F_{v_1})} f_{v_1}(gz) \chi_{v_1}(z) dz$$

is a matrix coefficient of a supercuspidal representation of $G(F_{v_1})$. In particular, we require that v_1 is non-archimedean.

- For at least one place $v_2 \neq v_1$, the test function f_{v_2} is supported on the locus of Z -regular semisimple elements of $G(F_{v_2})$. The place v_2 is not required to be non-archimedean.

For a cuspidal automorphic representation π of $G(\mathbb{A})$, we define a (global) spherical character as a distribution on $G(\mathbb{A})$:

$$(2.20) \quad J_\pi(f) = \sum_{\phi \in \mathcal{B}(\pi)} \left(\int_{H(F) \backslash H(\mathbb{A})} \pi(f)\phi(x) dx \right) \left(\int_{H(F) \backslash H(\mathbb{A})} \overline{\phi(x)} dx \right),$$

where the sum is over an orthonormal basis $\mathcal{B}(\pi)$ of π .

We now ready to state a simple RTF for nice test functions on $G(\mathbb{A})$:

Theorem 2.4. *Let χ be a (unitary) character of $Z_G(F) \backslash Z_G(\mathbb{A})$. If f is a nice test function with respect to χ , then $J_\chi(f)$ is equal to*

$$\sum_{\delta} O_\chi(\delta, f) = \sum_{\pi} J_\pi(f),$$

where the sum in left hand side runs over all regular semisimple orbits

$$\delta \in H(F) \backslash G(F) / Z_G(F) H(F),$$

and the right hand side runs over all cuspidal automorphic representations π with central character χ .

Here in the right hand side, by a π we mean a sub-representation of the space of cuspidal automorphic forms. So, a priori, two such representations may be isomorphic (as we don't know yet the multiplicity one for such a π , which is expected to hold by the Langlands–Arthur classification).

Proof. The proof follows the same line as that of Theorem 2.3 in the general linear case. \square

2.4 Comparison: fundamental lemma and transfer

Smooth transfer. We first recall the matching of orbits without proof. The proof can be found [51] and [62, §2.1]. Now the field F is either a number field or a local field of characteristic zero. We will view both S_{n+1} and $U(V)$ as closed subvarieties of $\text{Res}_{E/F} \text{GL}_{n+1}$. In the case of $U(V)$, this depends on a choice of an E -basis of V . Even though such choice is not unique, the following notion is independent of the choice: we say that $\delta \in U(V)(F)$ and $s \in S_{n+1}(F)$ match if s and δ (both considered as elements in $\text{GL}_{n+1}(E)$) are conjugate by an element in $\text{GL}_n(E)$. Then it is proved in [62, §2] that this defines a natural bijection between the set of regular semisimple orbits of $S_{n+1}(F)$ and the disjoint union of regular semisimple orbits of $U(V)$ where $V = W \oplus Eu$ (with $(u, u) = 1$) and W runs over all (isometric classes of) Hermitian spaces over E .

Now let E/F be number fields. To state the matching of test functions, we need to introduce a “transfer factor”: it is a compatible family of functions $\{\Omega_v\}_v$ indexed by all places v of F , where Ω_v is defined on the regular semisimple locus of $S_{n+1}(F_v)$, and they satisfy:

- If $s \in S_{n+1}(F)$ is regular semisimple, then we have a product formula

$$\prod_v \Omega_v(s) = 1.$$

- For any $h \in \mathrm{GL}_n(F_v)$ and $s \in S_{n+1}(F_v)$, we have $\Omega_v(h^{-1}sh) = \eta(h)\Omega_v(s)$.

The transfer factor is not unique. But we may construct one as follows. We have fixed a character $\eta' : E^\times \backslash \mathbb{A}_E^\times \rightarrow \mathbb{C}^\times$ (not necessarily quadratic) such that its restriction $\eta'|_{\mathbb{A}^\times} = \eta$. We define

$$(2.21) \quad \Omega_v(s) := \eta'_v(\det(s)^{-[(n+1)/2]} \det(e, es, \dots, es^n)).$$

Here $e = e_{n+1} = (0, \dots, 0, 1)$ and (e, es, \dots, es^n) is the $(n+1) \times (n+1)$ -matrix whose i -th row is es^{i-1} . It is easy to verify that such a family $\{\Omega_v\}_v$ defines a transfer factor.

We also extend this to a transfer factor on G' , by which we mean a compatible family of functions (to abuse notation) $\{\Omega_v\}_v$ on the regular semisimple locus of $G'(F_v)$, indexed by all places v of F , such that

- If $\gamma \in G'(F)$ is regular semisimple, then we have a product formula

$$\prod_v \Omega_v(\gamma) = 1.$$

- For any $h_i \in H'_i(F_v)$ and $\gamma \in G'(F_v)$, we have $\Omega_v(h_1\gamma h_2) = \eta(h_2)\Omega_v(\gamma)$.

We may construct it as follows. Write $\gamma = (\gamma_1, \gamma_2) \in G'(F_v)$ and $s = \nu(\gamma_1\gamma_2^{-1}) \in S_{n+1}(F_v)$. If n is odd, we set:

$$(2.22) \quad \Omega_v(\gamma) := \eta'_v(\det(\gamma_1\gamma_2^{-1}))\eta'_v(\det(s)^{-(n+1)/2} \det(e, es, \dots, es^n)),$$

and if n is even, we set:

$$(2.23) \quad \Omega_v(\gamma) := \eta'_v(\det(s)^{-n/2} \det(e, es, \dots, es^n)).$$

For a place v of F , we consider $f' \in \mathcal{C}_c^\infty(S_{n+1}(F_v))$ and the tuple $(f_W)_W, f_W \in \mathcal{C}_c^\infty(U(V)(F_v))$ indexed by the set of all (isometric classes of) Hermitian spaces W over $E_v = E \otimes F_v$, where we set $V = W \oplus E_v u$ with $(u, u) = 1$ as in (1.3). In particular, V is determined by W . We say that $f' \in \mathcal{C}(S_n(F_v))$ and the tuple $(f_W)_W$ are (smooth) transfer of each other if

$$\Omega_v(s)O(s, f') = O(\delta, f_W),$$

whenever a regular semisimple $s \in S_{n+1}(F_v)$ matches a $\delta \in U(V)(F_v)$.

Similarly we extend the definition to (smooth) transfer between elements in $\mathcal{C}_c^\infty(G'(F_v))$ and those in $\mathcal{C}_c^\infty(G^W(F_v))$, where we use G^W as in (1.4) to indicate the dependence on W . It is then obvious that the existence of the two transfers are equivalent. Similarly, we may extend the definition of (smooth) transfer to test functions on $G'(\mathbb{A})$ and $G^W(\mathbb{A})$.

For a split place v , the existence of transfer is almost trivial. To see this, we may directly work with smooth transfer on $G'(F_v)$ and $G^W(F_v)$. We may identify $\mathrm{GL}_n(E \otimes F_v) = \mathrm{GL}_n(F_v) \times \mathrm{GL}_n(F_v)$ and write the function $f'_n = f'_{n,1} \otimes f'_{n,2} \in \mathcal{C}_c^\infty(\mathrm{GL}_n(E \otimes F_v))$. There is only one isometric class of Hermitian space W for E_v/F_v . We identify the unitary group $U(W)(F_v) \simeq \mathrm{GL}_n(F_v)$ and let $f_n \in \mathcal{C}_c^\infty(U(W)(F_v)) = \mathcal{C}_c^\infty(\mathrm{GL}_n(F_v))$. Similarly we have f'_{n+1} for $\mathrm{GL}_{n+1}(F_v)$ etc..

Proposition 2.5. *If v is split in E/F , then the smooth transfer exists. In fact we may take the convolution $f_i = f'_{i,1} \star f'_{i,2}{}^\vee$ where $i = n, n+1$ and $f'_{i,2}{}^\vee(g) = f'_{i,2}(g^{-1})$*

Proof. In this case the quadratic character η_v is trivial. For $f' = f'_{n+1} \otimes f'_n$, the orbital integral $O(\gamma, f')$ can be computed in two steps: first we integrate over $H_2(F_v)$ then over the rest. Define

$$\bar{f}'_i(x) = \int_{\mathrm{GL}_i(F_v)} f'_{i,1}(xy) f'_{i,2}(y) dy = f'_{i,1} \star f'_{i,2}{}^\vee(x), \quad i = n, n+1.$$

Then obviously we have the orbital integral for $\gamma = (\gamma_{n+1}, \gamma_n) \in G'(F_v)$ and $\gamma_i = (\gamma_{i,1}, \gamma_{i,2}) \in \mathrm{GL}_i(F_v) \times \mathrm{GL}_i(F_v)$, $i = n, n+1$:

$$O(\gamma, f') = \int_{\mathrm{GL}_{n+1}(F_v)} \int_{\mathrm{GL}_n(F_v)} \bar{f}'_{n+1}(x\gamma_{n+1,1}\gamma_{n+1,2}^{-1}y) \bar{f}'_n(x\gamma_{n,1}\gamma_{n,2}^{-1}y) dx dy.$$

Now the lemma follows easily. □

Now use E/F to denote a local (genuine) quadratic field extension. We write Ω for the local transfer factor defined by (2.22) and (2.22). The main local result of this paper is the following:

Theorem 2.6. *If E/F is non-archimedean, then the smooth transfer exists.*

The proof will occupy section 3 and 4.

Let χ be a character of $Z_G(F)$ and define the character χ' of $Z_{G'}(F)$ to be the base change of χ .

Corollary 2.7. *If f' and f_W match, then the χ -orbital integrals also match, i.e.:*

$$O_\chi(\delta, f_W) = \Omega(\gamma) O_{\chi'}(\gamma, f')$$

whenever γ and δ match.

Proof. It suffices to verify that the orbital integrals are compatible with multiplication by central elements in the following sense: consider $z \in E^\times \times E^\times$ identified with the center of $G'(F)$ in the obvious way. We denote by \bar{z} the Galois conjugate coordinate-wise. Then $z/\bar{z} \in E^1 \times E^1$ which can be identified with the center of $G(F)$ in the obvious way. Assume that δ and γ match. Then so do $z\gamma$ and $z/\bar{z}\delta$. We have by assumption that f' and f_W match:

$$\Omega(z\gamma) O(z\gamma, f') = O(z/\bar{z}\delta, f_W)$$

for all z . It is an easy computation to show that our definition of transfer factors satisfy

$$\Omega(z\gamma) = \Omega(\gamma).$$

□

Fundamental lemma. We will need the fundamental lemma for units in the spherical Hecke algebras. Let E/F be an unramified quadratic extension (non-archimedean). There are precisely two isometric classes of Hermitian space W : one with a self-dual lattice is denoted by W_0 and the other W_1 . For W_0 , the Hermitian space $V = W \oplus Eu$ with $(u, u) = 1$ also has a self-dual lattice. We denote by K the subgroup of G^{W_0} which is the stabilizer of the self-dual lattice. Denote by K' the maximal subgroup $G'(\mathcal{O}_F)$ of $G'(F)$. Denote by 1_K and $1_{K'}$ the corresponding characteristic function. Choose measures on $G^{W_0}(F), G'(F)$ so that the volume of K, K' are all equal to one.

Theorem 2.8. *There is a constant $c(n)$ depending only on n such that the fundamental lemma of Jacquet–Rallis holds for all quadratic extension E/F with residue character larger than $c(n)$; namely, the function $1_K \in \mathcal{C}_c^\infty(G'(F))$ and the pair $f_{W_0} = 1_{K'}, f_{W_1} = 0$ are transfer of each other.*

Proof. This is proved in [61] by Z. Yun in the positive characteristic case, extended to characteristic zero by J. Gordon in the appendix to [61]. \square

An automorphic-Cebotarev-density theorem. We will need a theorem of *automorphic-Cebotarev-density* type proved by Ramakrishnan. It will allow us to separate (cuspidal) spectrums without using the fundamental lemma for the full spherical Hecke algebras at non-split places. It is stronger than the strong multiplicity one theorem for GL_n .

Theorem 2.9. *Let E/F be a quadratic extension. Two cuspidal automorphic representations Π_1, Π_2 of $\mathrm{Res}_{E/F}\mathrm{GL}_n(\mathbb{A})$ are isomorphic if and only if $\Pi_{1,v} \simeq \Pi_{2,v}$ for almost all places v of F that are split in E/F .*

The proof can be found in [52].

The trace formula identity. We first have the following coarse form of a trace formula identity.

Proposition 2.10. *Fix a character χ of $Z_G(F)\backslash Z_G(\mathbb{A})$ and let χ' be its base change. Fix a split place v_0 and a supercuspidal representation π_{v_0} of $G(F_{v_0})$ with central character χ_{v_0} . Suppose that*

- f' and $(f_W)_W$ are nice test functions and are smooth transfer of each other.
- Let Π_{v_0} be the local base change of π_{v_0} . Then f'_{v_0} is essentially a matrix coefficient of Π_{v_0} and is related to f_{W,v_0} as prescribed by Proposition 2.5 (in particular, f_{W,v_0} is essentially a matrix coefficient of π_{v_0}).

Fix a representation $\otimes_v \pi_v^0$ where the product is over almost all split places v and each π_v^0 is irreducible unramified. Then we have

$$\sum_{\Pi} I_{\Pi}(f') = \sum_W \sum_{\pi_W} J_{\pi_W}(f_W),$$

where the sums run over all automorphic representations Π of $G'(\mathbb{A})$ and π_W of $G^W(\mathbb{A})$ with central characters χ', χ respectively such that

- $\pi_{W,v} \simeq \pi_v^0$ for almost all split v .
- π_{W,v_0} is the fixed supercuspidal representation π_{v_0} .
- $\Pi = BC(\pi_W)$ is a weak base change of π_W , and Π_{v_0} is the local base change of π_{v_0} . In particular Π is cuspidal and the left hand side contains at most one term.

Proof. We may assume that all test functions are decomposable. Let S be a finite set of places such that

- all Hermitian spaces W with $f_W \neq 0$ are unramified outside S .
- for any v outside S , f'_v and $f_{W,v}$ are units of the spherical Hecke algebras (in particular, v is non-archimedean and unramified in E/F)

So we may identify $G^W(\mathbb{A}^S)$ (and write it as $G(\mathbb{A}^S)$) for all such W appeared in the sum. Now we enlarge S so that for all non-split v outside S , the fundamental lemma for units holds (Theorem 2.8). The fundamental lemma for the entire spherical Hecke algebra holds at all non-archimedean split places. Consider the spherical Hecke algebra $\mathcal{H}(G'(\mathbb{A}^S)//K'^S)$ where $K'^S = \prod_{v \notin S} K'_v$ is the usual maximal compact subgroup of $G'(\mathbb{A}^S)$, and the counterpart $\mathcal{H}(G(\mathbb{A}^S)//K^S)$ for unitary groups. For any $f'^S \in \mathcal{H}(G'(\mathbb{A}^S)//K'^S)$ and $f^S \in \mathcal{H}(G(\mathbb{A}^S)//K^S)$ such that at a non-split $v \notin S$, f'_v, f_v are the units, we have a trace formula identity:

$$I(f'_S \otimes f'^S) = \sum_W J(f_{W,S} \otimes f^S).$$

Again all these test functions are nice so we may apply the simple trace formulae of Theorem 2.3 and 2.4:

$$\sum_{\Pi} I_{\Pi}(f'_S \otimes f'^S) = \sum_W \sum_{\pi_W} J_{\pi_W}(f_{W,S} \otimes f^S).$$

Here all Π, π_W are cuspidal automorphic representations whose component at v_0 are the given ones. Let λ_{Π^S} ($\lambda_{\pi_W^S}$, resp.) be the linear functional of the spherical Hecke algebras $\mathcal{H}(G'(\mathbb{A}^S)//K'^S)$ ($\mathcal{H}(G(\mathbb{A}^S)//K^S)$, resp.). Then we observe that

$$I_{\Pi}(f'_S \otimes f'^S) = \lambda_{\Pi^S}(f'^S) I_{\Pi}(f'_S \otimes 1_{K'^S})$$

and similarly for $J_{\pi_W}(f_{W,S} \otimes f^S)$. Note that we are only allowed to take the unit elements in the spherical Hecke algebras at almost all non-split spaces. Therefore we can view both sides as linear functionals on the spherical Hecke algebra $\mathcal{H}(G'(\mathbb{A}^{S,split})//K'^{S,split})$ where “split” indicate we only consider the product over all split places outside S . These linear functionals are linearly independent. In particular, for the fixed $\otimes_v \pi_v^0$, we may have an equality as claimed in the theorem. Since such Π 's are cuspidal, there exists at most one Π by Theorem 2.9. \square

Now we come to the trace formula identity which will allow us to deduce the main theorems in the introduction.

Proposition 2.11. *Let E/F be a quadratic extension such that all archimedean places $v|\infty$ are split. Fix a Hermitian space W_0 and define V_0 , the group $G = G^{W_0}$ by (1.3) and (1.4). Let π be a cuspidal automorphic representation of G such that for a split place v_0 , π_{v_0} is supercuspidal. Consider decomposable nice functions f' and $(f'_W)_W$ satisfying the same conditions as in Proposition 2.10. Then we have a trace formula identity:*

$$I_{\Pi}(f') = \sum_W \sum_{\pi_W} J_{\pi_W}(f_W),$$

where $\Pi = BC(\pi)$ and the sum in the right hand side runs over all W and all π_W nearly equivalent to π .

Proof. Apply Proposition 2.10 to $\pi_v^0 = \pi_v$ for almost all split v . Then in the sum of the right hand side there, all π_W have the same weak base change Π . Note that the local base change map are injective for split places and for unramified representations at non-split unramified places. By our Hypothesis (*), this implies that all π_W are in the same nearly equivalence class. □

A non-vanishing result. To see that the second condition in Theorem 1.1 on the niceness of a test function does not lose generality in some sense, at least for tempered representations at v , we will need some “regularity” result for the distribution J_{π} defined by (2.20). By the multiplicity one result of [3] and [55], we have $\dim \text{Hom}_{H_v}(\pi_v, \mathbb{C}) \leq 1$. We may fix an appropriate choice of generator $\ell_{H_v} \in \text{Hom}_{H_v}(\pi_v, \mathbb{C})$ ($\ell_{H_v} = 0$ if the space is zero) and decompose

$$(2.24) \quad \ell_H = c_{\pi} \prod_v \ell_{H_v},$$

where c_{π} is a constant depending on the cuspidal automorphic representation π (and its realization in $\mathcal{A}_0(G)$). This gives a decomposition of the spherical character as a product of local spherical characters

$$(2.25) \quad J_{\pi}(f) = |c_{\pi}|^2 \prod_v J_{\pi_v}(f_v),$$

where the spherical character is defined as

$$J_{\pi_v}(f_v) = \sum_{\phi_v \in \mathcal{B}(\pi_v)} \ell_{H_v}(\pi_v(f_v)\phi_v) \overline{\ell_{H_v}(\phi_v)}.$$

Note that J_{π_v} is a distribution of positive type, namely, for all $f_v \in \mathcal{C}_c^{\infty}(G(F_v))$,

$$J_{\pi_v}(f_v * f_v^*) \geq 0, \quad f_v^*(g) := \overline{f_v(g^{-1})}.$$

To see the positivity, we notice that

$$J_{\pi_v}(f_v * f_v^*) = \sum_{\phi_v \in \mathcal{B}(\pi_v)} \ell_{H_v}(\pi_v(f_v)\phi_v) \overline{\ell_{H_v}(\pi_v(f_v)\phi_v)} \geq 0.$$

We will also say that a function of the form $f_v * f_v^*$ is of positive type.

Proposition 2.12. *Let π_v be a tempered representation of $G(F_v)$. Then there exists function $f_v \in \mathcal{C}_c^\infty(G(F_v))$ supported in the Z -regular semisimple locus such that*

$$J_{\pi_v}(f_v) \neq 0.$$

The proof is given in the appendix A to this paper (Theorem A.2). Equivalently, the result can be stated as follows: the support of the spherical character J_{π_v} (as a distribution on $G(F_v)$) is not contained in the complement of Z -regular semisimple locus.

2.5 Proof of Theorem 1.1: (ii) \implies (i)

Proposition 2.13. *Let E/F be a quadratic extension of number fields such that all archimedean places $v|\infty$ are split. Let $W \subset V$ and $H \subset G$ be defined by (1.3) and (1.4). Let π be a cuspidal automorphic representation of G such that for a split place v_1 , π_{v_1} is supercuspidal and for a split place $v_2 \neq v_1$, π_{v_2} is tempered. Denote by $\Pi = BC(\pi)$ its weak base change.*

If π is distinguished by H , then $L(1/2, \Pi, R) \neq 0$ and Π is η -distinguished by $H'_2 = \mathrm{GL}_{n+1, F} \times \mathrm{GL}_{n, F}$. In particular, in Theorem 1.1, (ii) implies (i).

Proof. We apply Proposition 2.11. It suffices to show that there exist f' as in Proposition 2.11 such that

$$I_\Pi(f') \neq 0.$$

We will first choose an appropriate $f := f_W$ and then choose f' to be a transfer of the tuple $(f_W, 0, \dots, 0)$ where for all hermitian space other than W we choose the zero functions. We choose f' satisfying the conditions of Proposition 2.11. Then the trace formula identity from Proposition 2.11 is reduced to

$$I_\Pi(f') = \sum_{\pi_W} J_{\pi_W}(f_W).$$

Note that for all π_W , they have the same local component at v_1, v_2 by our Hypothesis (*) on the local-global compatibility for weak base change at split places.

We choose $f = f_W = \otimes_v f_v$ as follows. By the assumption on the distinction of π , we have $c_\pi \neq 0$ in (2.24) and we may choose a function $g = \otimes_v g_v$ of positive type on $G^W(\mathbb{A})$ such that $J_{\pi_v}(g_v) > 0$ for all v . We may assume that at v_1 , g_{v_1} is essentially a matrix coefficient of π_{v_1} . This is clearly possible. Then we have

$$J_\pi(g) = |c_\pi|^2 \prod_v J_{\pi_v}(g_v) > 0$$

and for all π_W nearly equivalent to π :

$$J_{\pi_W}(g) \geq 0.$$

Now we choose $f_v = g_v$ for every place v other than v_2 . We choose f_{v_2} to be supported in the Z -regular semisimple locus. By Proposition 2.12, we may choose an f_{v_2} such that

$$J_{\pi_{v_2}}(f_{v_2}) \neq 0.$$

For this choice of f , the trace identity is reduced to

$$I_{\Pi}(f') = J_{\pi_{v_2}}(f_{v_2}) \left(\sum_{\pi_W} |c_{\pi_W}|^2 J_{\pi_W^{(v_2)}}(f^{(v_2)}) \right)$$

where the superscript indicates the away from v_2 -part: $J_{\pi_W^{(v_2)}} = \prod_{v \neq v_2} J_{\pi_{W,v}}$. In the sum, every term is non-negative as we choose $f^{(v_2)} = g^{(v_2)}$ of positive type. And at least one of these terms (the one from π) is non-zero. Therefore we conclude that for this choice the right hand side above is non-zero. This shows that $I_{\Pi}(f') \neq 0$ and completes the proof. Note that the proof should be much easier if we assume the multiplicity one for π_W in the cuspidal spectrum of the unitary group. \square

2.6 Proof of Theorem 1.4

A key ingredient is the following Burger–Sarnak type principle à la Prasad [47].

Proposition 2.14. *Let V be a Hermitian space of dimension $n + 1$ and W a nondegenerate subspace of codimension one. Let π be a cuspidal automorphic representation of $U(V)(\mathbb{A})$. Fix a finite (non-empty) set S of places and an irreducible representation σ_v of $U(W)(F_v)$ for each $v \in S$ such that*

- *If $v \in S$ is archimedean, both W and V are positive definite at v .*
- *If $v \in S$ is non-archimedean and split, σ_v^0 is induced from a representation of $Z_v K_v$ where K_v is a compact open subgroup and Z_v is the center of $U(W)(F_v)$.*
- *If $v \in S$ is either archimedean or non-split, the contragredient of σ_v^0 appears as a quotient of π_v restricted to $U(W)(F_v)$.*

Then there exists a cuspidal automorphic representation σ of $U(W)(\mathbb{A})$ such that

- *$\sigma_v = \sigma_v^0$ for all $v \in S$.*
- *the linear form ℓ_W on $\pi \otimes \sigma$ is non-zero.*

Heuristically, this allows to pair π with a σ with prescribed local components at S such that $\pi \otimes \sigma$ is distinguished.

We first show the following variant of [47, Lemma 1]. Note that the only difference lies in the assumption on the center. The assumption on the center seems to be indispensable. For example, it seems to be difficult to prove the same result if $G = \mathrm{GL}_{n+1}$ and $H = \mathrm{GL}_n$.

Lemma 2.15. *Suppose that we are in the following situation:*

- *F is a number field.*
- *G is a reductive algebraic group defined over F , and H is a reductive subgroup of G .*
- *S is a finite set of places of F such that: if $v \in S$ is archimedean, then $G(F_v)$ is compact. Denote $G_S = \prod_{v \in S} G(F_v)$ and $H_S = \prod_{v \in S} H(F_v)$.*

- The center Z of H is anisotropic over F .

Let π be a cuspidal automorphic representation of $G(\mathbb{A})$. Let $\otimes_{v \in S} \mu_v$ be an irreducible representation of H_S such that

- (1) For each $v \in S$, μ_v appears as a quotient of π_v restricted to $H(F_v)$.
- (2) for each non-archimedean $v \in S$, μ_v is supercuspidal representations of $H(F_v)$, and it is an induced representation $\mu_v = \text{Ind}_{Z_v K_v}^{H_v} \nu_v$ from a representation ν_v of a subgroup $Z_v K_v$, where K_v is an open compact subgroup of H_v .

Then there is an automorphic representation $\mu' = \prod_v \mu'_v$ of $H(\mathbb{A})$ and functions $f_1 \in \pi$, $f_2 \in \mu'$ such that

(i)

$$\int_{H(F) \backslash H(\mathbb{A})} f_1(h) \bar{f}_2(h) dh \neq 0,$$

and

- (ii) If $v \in S$ is archimedean $\mu'_v = \mu_v$; if $v \in S$ is non-archimedean, $\mu'_v = \text{Ind}_{Z_v K_v}^{H_v} \nu'_v$ is induced from ν'_v where $\nu'_v|_{K_v} = \nu_v|_{K_v}$.

Proof. The proof is a variant of [47, Lemma 1]. If $v \in S$ is archimedean, let $K_v = H(F_v)$ and $\nu_v = \mu_v$. It is compact by assumption. We consider the restriction of π_v to K_v for each $v \in S$. By the assumption and Frobenius reciprocity, $\nu_v|_{K_v}$ is a quotient representation of $\pi_v|_{K_v}$. Since K_v is compact, $\nu_v|_{K_v}$ is also a sub-representation for $v \in S$. This means that we may find a function f on $G(\mathbb{A})$ whose $K_S = \prod_{v \in S} K_v$ translates span a space which is isomorphic to $\otimes_{v \in S} \nu_v|_{K_v}$ as K_S -modules. By the same argument as in [47, Lemma 1] (using weak approximation), we may assume that such f has non-zero restriction (denoted by \tilde{f}) to $H(F) \backslash H(\mathbb{A})$. Now note that $Z(F) \backslash Z(\mathbb{A})$ is compact. For a character χ of $Z(F) \backslash Z(\mathbb{A})$ we may define

$$\tilde{f}_\chi(h) := \int_{Z(F) \backslash Z(\mathbb{A})} f(zh) \chi^{-1}(z) dz, \quad h \in H(F) \backslash H(\mathbb{A}).$$

As Z_S and K_S commute, each of \tilde{f} and \tilde{f}_χ generates a space of functions on $H(F) \backslash H(\mathbb{A})$ which is isomorphic to $\otimes_{v \in S} \nu_v|_{K_v}$ as K_S -modules. There must exist some χ such that it is non-zero. For such a χ , it is necessarily true that $\chi_v|_{Z_v \cap K_v} = \omega_{\nu_v}|_{Z_v \cap K_v}$ where ω_{ν_v} is the central character of ν_v . In particular, we may replace $\mu_v = \text{Ind}_{Z_v K_v}^{H_v} \nu_v$ by $\mu'_v := \text{Ind}_{Z_v K_v}^{H_v} \nu'_v$ where ν_v is an irreducible representation of $Z_v K_v$ with central character χ_v and $\nu'_v|_{K_v} = \nu_v|_{K_v}$. Certainly such μ'_v is still supercuspidal if $v \in S$ is non-archimedean. If $v \in S$ is archimedean, we have $\mu'_v = \mu_v$. Now we consider the space generated by \tilde{f}_χ under $Z_S K_S$ translations. This space is certainly isomorphic to $\prod_{v \in S} \nu'_v$ as $Z_S K_S$ -modules. The rest of the proof is the same as in [47, Lemma 1], namely applying [47, Lemma 2] to the space of H_S -translations of \tilde{f}_χ which is isomorphic to $\otimes_{v \in S} \text{Ind}_{Z_v K_v}^{H_v} \nu'_v$ as H_S -modules. \square

We now return to prove Proposition 2.14 .

Proof. We apply Lemma 2.15 above to $H = U(W), G = U(V)$. Then the center of H is anisotropic. If $v \in S$ is split non-archimedean, it is always true that μ_v appears as a quotient of π_v (the local conjecture in [14] for the general linear group is known to hold for generic representations). The proposition then follows immediately. \square

Remark 9. Noting that any supercuspidal representation of $\mathrm{GL}_n(F_v)$ for a non-archimedean local field F_v is induced from an irreducible representation of an open subgroup that is compact modulo center. But an open subgroup of $\mathrm{GL}_n(F)$ compact modulo center is not necessarily of the form $K_v Z_v$. As pointed out by Prasad to the author, it may be possible to choose an arbitrary supercuspidal μ_v if one suitably extends the result in Lemma 2.15.

Now we come to the proof of Theorem 1.4.

Proof. We show this by induction on the dimension $n = \dim W$ of W . If n is equal to one, then the theorem is obvious. Now assume that for all dimension at most n Hermitian spaces, the statement holds. Let V be a $n + 1$ -dimensional Hermitian space and W is a codimension one subspace. And let π be a cuspidal automorphic representation such that π_{v_1} is supercuspidal for a split place v_1 . By [5], there exists a supercuspidal representation $\sigma_{v_1}^0$ that verifies the assumptions of Proposition 2.14. Then we apply Proposition 2.14 to $S = \{v_1\}$ to choose a cuspidal automorphic representation σ of $U(W)$ such that $\pi \otimes \sigma$ is distinguished by H . Then by Proposition 2.13, the weak base change of $\pi \otimes \sigma$ is η -distinguished by $H'_2 = \mathrm{GL}_{n+1,F} \times \mathrm{GL}_{n,F}$. By induction hypothesis, the weak base change of σ is (η -, resp.) distinguished by $\mathrm{GL}_{n,F}$ if n is odd (even, resp.). Together we conclude that the weak base change of π is (η -, resp.) distinguished by $\mathrm{GL}_{n+1,F}$ if $n + 1$ is odd (even, resp.). This completes the proof. \square

Remark 10. If we have the trace formula identity for all test functions f (say, after one proves the fine spectral decomposition), then we may use the proof of Proposition 2.13 to show first the existence of weak base change, then use the proof of Theorem 1.4 to show the distinction of the weak base change as predicted by the conjecture of Flicker–Rallis. But it seems impossible to characterize the image of the weak base change using the Jacquet–Rallis trace formulae alone.

2.7 Proof of Theorem 1.1: (i) \implies (ii)

Now we finish the proof of the other direction of Theorem 1.1: (i) \implies (ii). We may prove a slightly stronger result, replacing the condition (2) by the following: “ π_{v_1} is supercuspidal at a split place v_1 , and π_{v_2} is tempered at a split place $v_2 \neq v_1$.”

By Theorem 1.4 (whose proof also works if we only assume the temperedness of π_{v_2}), the weak base change $\Pi = BC(\pi)$ is η -distinguished by H'_2 . By the assumption on nonvanishing of $L(1/2, \Pi, R)$, we know that Π is also distinguished by $\mathrm{Res}_{E/F} \mathrm{GL}_n$. Therefore, I_Π is a non-zero distribution on $G'(\mathbb{A})$ and we have $I_\Pi(f') \neq 0$ for some decomposable $f' = \otimes_v f'_v$. Note that the multiplicity one also holds in this case:

$$\dim \mathrm{Hom}_{H'_i(F_v)}(\Pi_v, \mathbb{C}) \leq 1.$$

Similar to the decomposition of the distribution J_π (2.25), we may fix a decomposition

$$I_\Pi = c_\Pi \prod_v I_{\Pi_v}.$$

In particular, $c_\Pi \neq 0$ and for the f' above, $I_{\Pi_v}(f'_v) \neq 0$ for all v . We want to modify f' at the two places v_1, v_2 to apply Proposition 2.11.

It is easy to see that we may replace f'_{v_1} by essentially a matrix coefficient of π_{v_1} such that the non-vanishing $I_\Pi \neq 0$ remains. Now note that, at the split place v_2 , there is a nonzero constant c_v :

$$I_{\Pi_{v_2}}(f'_{v_2}) = c_v J_{\pi_{v_2}}(f_{v_2})$$

if Π_{v_2} is the local base change of π_{v_2} and f_{v_2} is the transfer of f'_{v_2} as prescribed by Proposition 2.5. By Proposition 2.12, $J_{\pi_{v_2}}(f_{v_2}) \neq 0$ for some f_{v_2} supported in Z -regular semisimple locus. Therefore, we may choose f'_{v_2} supported in Z -regular semisimple locus such that $I_{\Pi_{v_2}}(f'_{v_2}) \neq 0$.

Now we replace $f'_{v_i}, i = 1, 2$ by the new choices. Then we let the tuple (f_W) be a transfer of f' satisfying the conditions in Proposition 2.11. By the trace formula identity of Proposition 2.11, we have

$$I_\Pi(f') = \sum_{\pi_{W'}} J_{\pi_{W'}}(f_{W'})$$

where the sum in right hand side runs over all W' , and all $\pi_{W'}$ nearly equivalent to π . There must be at least one term $J_{\pi'}(f_{W'}) \neq 0$ for some W' . This completes the proof of Theorem 1.1.

2.8 Proof of Theorem 1.2.

Now we may prove Theorem 1.2. Assume that $\sigma = BC(\pi_1)$ for a cuspidal automorphic representation π_1 of some unitary group $U(V)$ where $\dim V = n+1$. Then by Proposition 2.14, we may find a cuspidal automorphic π_2 of $U(W)$ for a Hermitian subspace W of codimension one such that $\pi = \pi_1 \otimes \pi_2$ is distinguished by H . Let τ be the weak base change of π_2 . Then by Theorem 1.1, the Rankin–Selberg L -function $L(\sigma \times \tau, \frac{1}{2}) \neq 0$. This completes the proof.

3 Reduction steps

In this and the next section, we will prove the existence of smooth transfer at a non-archimedean non-split place (Theorem 2.6) as well as a partial result at an archimedean non-split place (Theorem 3.14).

In this section, we reduce the question to an analogue on “Lie algebras” (an infinitesimal version) and then to a local question around zero. Let F be a local field of characteristic zero. In this section, *both archimedean and non-archimedean local fields* are allowed. Let $E = F[\sqrt{\tau}]$ be a quadratic extension where $\tau \in F^\times$. We remind the reader that, even though our interest is in the genuine quadratic extension E/F , we may actually allow E to be split, namely, $\tau \in (F^\times)^2$.

3.1 Reduction to Lie algebras

Categorical quotients. We consider the action of $H' := \mathrm{GL}_{n,F}$ on the tangent space of the symmetric space S_{n+1} (cf. §2.1) at the identity matrix 1_{n+1} :

$$(3.1) \quad \mathfrak{S}_{n+1} := \{x \in M_{n+1}(E) \mid x + \bar{x} = 0\},$$

which will be called the ‘‘Lie algebra’’ of S_{n+1} . When no confusion arises, we will write it as \mathfrak{S} for simplicity. It will be more convenient to consider the action of H' on the Lie algebra of $\mathrm{GL}_{n+1,F}$:

$$\mathfrak{gl}_{n+1} \simeq \{x \in M_n(E) \mid x = \bar{x}\}.$$

The right hand side is isomorphic to \mathfrak{S}_{n+1} non-canonically.

Let W be a Hermitian space of dimension n and let $V = W \oplus Eu$ with $(u, u) = 1$. We identify the Lie algebra (as an F -vector space) of $U(V)$ with:

$$(3.2) \quad \mathfrak{U}(V) = \{x \in \mathrm{End}_E(V) \mid x + x^* = 0\},$$

where x^* is the adjoint of x with respect to the Hermitian form on V :

$$(xa, b) = (a, x^*b), \quad a, b \in V.$$

We consider the restriction to $H = H_W = U(W)$ of the adjoint action of $U(V)$ on $U(V)$ and $\mathfrak{U}(V)$.

Relative to the H -action or H' -action, we have notions of regular semisimple elements. Analogous to the group case, regular semisimple elements have trivial stabilizers. We also define an analogous matching of orbits as follows. We may identify $\mathrm{End}_E(V)$ with $M_{n+1}(E)$ by choosing a basis of V . Then for regular semisimple $x \in \mathfrak{S}(F)$ and $y \in \mathfrak{U}(V)(F)$, we say that they (and their orbits) match each other if x and y , considered as elements in $M_{n+1}(E)$, are conjugate by an element in $\mathrm{GL}_n(E)$. We will also say that x and y are transfer of each other and denote the relation by $x \leftrightarrow y$.

Then, analogous to the case for groups, the notion of transfer defines a bijection between regular semisimple orbits

$$(3.3) \quad \mathfrak{S}(F)_{rs}/H'(F) \simeq \coprod_W \mathfrak{U}(V)(F)_{rs}/H(F),$$

where the disjoint union runs over all isomorphism classes of n -dimensional Hermitian space W . We recall some results from [51], [62, §2]. For the natural map $\pi'_F : \mathfrak{S}(F) \rightarrow (\mathfrak{S}/H')(F)$ ($\pi_{W,F} : \mathfrak{U}(V)(F) \rightarrow (\mathfrak{U}(V)/H)(F)$, resp.), the fiber of a regular semisimple element consists of precisely one orbit (at most one, resp.) with trivial stabilizer. Moreover, π'_F is surjective. In particular, π'_F induces a bijection:

$$\mathfrak{S}(F)_{rs}/H'(F) \simeq (\mathfrak{S}/H')(F)_{rs}.$$

And $\pi_{W,F}$ induces a bijection between $\mathfrak{U}(V)(F)_{rs}/H(F)$ and its image in $(\mathfrak{U}(V)/H)(F)$.

A more intrinsic way is to establish an isomorphism between the categorical quotients between \mathfrak{S}/H' and $\mathfrak{U}(V)/H$. To state this more precisely, let us consider the invariants on them. We may choose a set of invariants on \mathfrak{S}_{n+1}

$$(3.4) \quad \text{tr } \wedge^i x, \quad e \cdot x^j \cdot e^*, \quad 1 \leq i \leq n+1, 1 \leq j \leq n;$$

and on $\mathfrak{U}(V)$ for $V = W \oplus Eu$:

$$(3.5) \quad \text{tr } \wedge^i y, \quad (y^j u, u), \quad 1 \leq i \leq n+1, 1 \leq j \leq n,$$

where $x \in \mathfrak{S}_{n+1}$ and $y \in \mathfrak{U}(V)$. If we write $\mathfrak{S}_{n+1} \ni x = \begin{pmatrix} A & b \\ c & d \end{pmatrix}$, $A \in \mathfrak{S}_n$, an equivalent set of invariants on \mathfrak{S}_{n+1} are

$$(3.6) \quad \text{tr } \wedge^i A, \quad c \cdot A^j \cdot b, \quad d, \quad 1 \leq i \leq n, 0 \leq j \leq n-1.$$

Similarly for the unitary case.

Denote by $Q = \mathbb{A}^{2n+1}$ the $2n+1$ -dimensional affine space (in this and the next section we are always in the local situation and \mathbb{A} will denote the affine line instead of the ring of adeles). Then the invariants above define a morphism

$$\begin{aligned} \pi_{\mathfrak{S}} : \mathfrak{S}_{n+1} &\rightarrow Q \\ x &\mapsto (\text{tr } \wedge^i x, e \cdot x^j \cdot e^*), \quad i = 1, 2, \dots, n+1, j = 1, 2, \dots, n. \end{aligned}$$

To abuse notation, we will also denote by $\pi_{\mathfrak{S}}$ the morphism defined by the second set of invariants above. Similarly we have morphism denoted by $\pi_{\mathfrak{U}}$ for the unitary case. We will simply write π if no confusion arises.

Lemma 3.1. *For each case $\mathcal{V} = \mathfrak{S}$ or $\mathfrak{U}(V)$, the pair $(Q, \pi_{\mathcal{V}})$ defines a categorical quotient of \mathcal{V} .*

Equivalently, the set of invariants defined by (3.4) ((3.5), resp.) is a set of generators of the ring of invariant polynomials on \mathfrak{S}_{n+1} ($\mathfrak{U}(V)$, resp.). Moreover, we have an obvious analogue if we replace \mathfrak{S}_{n+1} by \mathfrak{gl}_{n+1} .

Proof. As this is a geometric statement, we may extend the base field to the algebraic closure where two cases coincide. Hence it suffices to treat the case $\mathcal{V} = \mathfrak{S}$ or the equivalent case $\mathcal{V} = \mathfrak{gl}_{n+1}$. We will use the set of invariants (3.6) for \mathfrak{gl}_{n+1} . We will use Igusa's criterion ([33, Lemma 4], or [48, Theorem 4.13]): Let a reductive group H act on an irreducible affine variety \mathcal{V} . Let Q be a normal irreducible affine variety, and $\pi : \mathcal{V} \rightarrow Q$ be a morphism which is constant on the orbits of H such that

- (1) $Q - \pi(\mathcal{V})$ has codimension at least two;
- (2) There exists a non-empty open subset Q_0 of Q such that the fiber $\pi^{-1}(q)$ of $q \in Q_0$ contains exactly one closed orbit.

Then (Q, π) is a categorical quotient for the H -action on V .

For \mathfrak{gl}_{n+1} , the morphism π is clearly constant on the orbits of H . Now we define a section of π close to the classical companion matrices. Consider

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & 0 & 1 & 0 \\ b_n & b_{n-1} & \dots & b_1 & 1 \\ a_n & a_{n-1} & \dots & a_1 & d \end{pmatrix}.$$

Then its invariants are

$$\mathrm{tr} \wedge^i A = (-1)^{i-1} b_i, c \cdot A^j \cdot b = a_{j+1}, d \quad i = 1, 2, \dots, n, j = 0, 1, 2, \dots, n-1.$$

This gives us an explicit choice of section of π and it shows that π is surjective. This verifies (1). By [51], for all regular semisimple $q \in Q$, the fiber of q consists of at most one closed orbit. It follows by the explicit construction above that the fiber contains precisely one closed orbit. The regular semisimple elements form the complement of a principle divisor and hence we have verified condition (2). This completes the proof. \square

By this result, we have a natural isomorphism between the categorical quotients \mathfrak{S}/H' and $\mathfrak{U}(V)/H$. In the bijection (3.3) the appearance of disjoint union is due to the fact that the map between F -points induced by $\pi_{\mathfrak{S}}$ is surjective but the one induced by $\pi_{\mathfrak{U}(V)}$ is not.

Lemma 3.1 also allows us to transfer semisimple elements (not necessarily regular): we say that two *semisimple* elements $x \in \mathfrak{S}(F)$ and $y \in \mathfrak{U}(V)(F)$ match each other if they have the same invariants, or equivalently, their images in the quotients correspond to each other under the isomorphism between the categorical quotients. A warning is that, given a semisimple $x \in \mathfrak{S}(F)$ (not necessary regular), in general there may be more than one matching semisimple orbits in $\mathfrak{U}(V)(F)$.

Smooth transfer conjecture of Jacquet–Rallis. Before we state the infinitesimal version of smooth transfer, we need to define a transfer factor on the level of Lie algebras.

Definition 3.2. Consider the action of H' on $X = \mathfrak{gl}_{n+1}$ or \mathfrak{S} . A transfer factor is a smooth function $\omega : X(F)_{rs} \rightarrow \mathbb{C}^\times$ such that $\omega(x^h) = \eta(h)\omega(x)$.

Obviously, two transfer factors ω, ω' differer by a $H'(F)$ -invariant smooth function $\xi : X(F)_{rs} \rightarrow \mathbb{C}^\times$. If ξ extends to a smooth function on $X(F)$ (with moderate growth towards infinity for a norm on $X(F)$ if F is archimedean), we say that ω, ω' are equivalent and denote by $\omega \sim \omega'$.

We have fixed a transfer factor Ω earlier on the relevant groups by (2.21), (2.22) and (2.23). We now define a transfer factor on the Lie algebras. If $\sqrt{\tau}x \in \mathfrak{S}(F)$ is regular semisimple, we define

$$(3.7) \quad \omega(\sqrt{\tau}x) := \eta(\det(e, ex, ex^2, \dots, ex^n)).$$

Now we may similarly define the notion of transfer of test functions on $\mathfrak{S}(F)$ and $\mathfrak{U}(V)(F)$. For an $f' \in \mathcal{C}_c^\infty(\mathfrak{S}(F))$, and a tuple $(f_W)_W$ where $f_W \in \mathcal{C}_c^\infty(\mathfrak{U}(V)(F))$, they are called a (smooth) transfer of each other if for all matching regular semisimple $\mathfrak{S}(F) \ni x \leftrightarrow y \in \mathfrak{U}(V)(F)$, $V = W \oplus Eu$, we have

$$\omega(x)O(x, f') = O(y, f_W).$$

For $n \in \mathbb{Z}_{\geq 1}$, we rewrite the ‘‘smooth transfer conjecture’’ of Jacquet-Rallis ([39]) for the symmetric space S_{n+1} and the unitary group $U(V)$ as follows:

Conjecture S_{n+1} : For any $f' \in \mathcal{C}_c^\infty(S_{n+1}(F))$, its transfer $(f_W)_W$ exists, where $f_W \in \mathcal{C}_c^\infty(U(V)(F))$. And the other direction also holds, namely, given any a tuple $(f_W)_W$, there exists its transfer f' .

The corresponding statement for Lie algebras can be stated as

Conjecture \mathfrak{S}_{n+1} : For any $f' \in \mathcal{C}_c^\infty(\mathfrak{S}_{n+1}(F))$, its transfer $(f_W)_W$ exists, where $f_W \in \mathcal{C}_c^\infty(\mathfrak{U}(V)(F))$. And the other direction also holds, namely, given any a tuple $(f_W)_W$, there exists its transfer f' .

Note that the statement depends on the choice of a transfer factor. But it is obvious that the truth of the conjecture does not depend on the choice of the transfer factor within an equivalence class.

Reduction to Lie algebras. We now reduce the group version of smooth transfer conjecture to the Lie algebra one.

Theorem 3.3. Conjecture \mathfrak{S}_{n+1} implies Conjecture S_{n+1} .

To prove this theorem, we need some preparation. For $\nu \in E$ we define a set

$$D_\nu = \{x \in M_{n+1}(E) \mid \det(\nu - x) = 0\}.$$

We will choose a basis of V to realize the unitary group $U(V)$ ($\mathfrak{U}(V)$, resp.) as a subgroup (a F -sub-vector-space, resp.) of $GL_{n+1, E}$ ($M_{n+1}(E)$, resp.).

Lemma 3.4. Let $\xi \in E^1$. The Cayley map

$$\begin{aligned} \alpha_\xi : M_{n+1}(E) - D_1 &\rightarrow GL_{n+1}(E) \\ x &\mapsto -\xi(1+x)(1-x)^{-1}. \end{aligned}$$

induces an H -equivariant isomorphism between $\mathfrak{S}_{n+1}(F) - D_1$ and $S_{n+1}(F) - D_\xi$. In particular, if we choose a sequence of distinct $\xi_1, \xi_2, \dots, \xi_{n+2} \in E^1$, the images of $\mathfrak{S}_{n+1}(F) - D_1$ under α_{ξ_i} form a finite cover by open subset of $S_{n+1}(F)$.

Similarly, α_ξ induces an $U(W)$ -equivariant isomorphism between $\mathfrak{U}(V)(F) - D_1$ and $U(V)(F) - D_\xi$.

Proof. First it is easy to verify that the image of α_ξ lies in $S = S_{n+1}$, i.e.,:

$$\alpha(x)\bar{\alpha}(x) = (1+x)(1-x)^{-1}(1+\bar{x})(1-\bar{x})^{-1} = 1,$$

which holds as long as $\text{char } F \neq 2$. Now we note that if $\det(\xi - s) \neq 0$, α_ξ has an inverse defined by

$$s \mapsto -(\xi + s)(\xi - s)^{-1}.$$

This shows that the image of α_ξ is $S - D_\xi$ and α_ξ defines an isomorphism between two affine varieties.

The same argument proves the desired assertion in the unitary case. \square

Lemma 3.5. *The transfer factors are compatible under the Cayley map α_μ .*

Proof. It suffices to consider the case $\mu = 1$ as the argument is the same for a general μ . Note that $(1 + x)$ and $(1 - x)^{\pm 1}$ commute. We have

$$\Omega((1 + x)(1 - x)^{-1}) = \eta'(\det((1 + 2(1 - x)^{-1})^i e^*)_{i=0}^{n-1}).$$

Set $T = 2(1 - x)^{-1}$. Then it is easy to see that the determinant is equal to

$$\det((1 + T)^i e^*)_{i=0}^{n-1} = \det(T^i e^*)_{i=0}^{n-1}$$

by elementary operations on a matrix. This is equal to

$$2^n (-1)^{n(n-1)/2} \det((1 - x)^i e^*)_{i=0}^{n-1} = 2^n \det(x^i e^*)_{i=0}^{n-1}.$$

Therefore we have proved that

$$\Omega((1 + x)(1 - x)^{-1}) = \eta'(2^n \det(x^i e^*)_{i=0}^{n-1}) = c \cdot \omega(x/\sqrt{\tau}).$$

for a non-zero constant c . \square

For more flexibility, we will consider the following statement P_μ indexed by $\mu \in F^\times$:

P_μ : For $f \in \mathcal{C}_c^\infty(\mathfrak{S}_{n+1} - D_\mu)$, its transfer (f_W) exists and can be chosen such that $f_W \in \mathcal{C}_c^\infty(\mathfrak{U}(V) - D_\mu)$. And the other direction also holds.

Then it is clear that if P_ν holds for all $\mu \in F^\times$, then Conjecture S_{n+1} follows by applying a partition of unity argument to the open cover of S_{n+1} and $U(V)$ for distinct μ_0, \dots, μ_{n+1} .

To prove Theorem 3.3, it remains to show the following:

Lemma 3.6. *Conjecture \mathfrak{S}_{n+1} implies P_μ for all $\mu \in F^\times$.*

Proof. Fix a μ . Assume that (f_W) is a transfer of $f \in \mathcal{C}_c^\infty(\mathfrak{S} - D_\mu)$. Let $Y = \text{supp}(f) \subseteq \mathfrak{S}(F)$ and $Z = \pi'(Y)$. It suffices to show that for each W there exists a function $\alpha_W \in \mathcal{C}^\infty(\mathfrak{U}(V)(F))$ (smooth when F is archimedean, locally constant when F is non-archimedean) satisfying

1. α_W is $H(F)$ -invariant,
2. $\alpha_W|_{\pi_W^{-1}(Z)} \equiv 1$,
3. $\alpha_W|_{D_\mu} \equiv 0$.

Then we may replace f_W by $f_W\alpha_W$, which will still be in $\mathcal{C}_c^\infty(\mathfrak{U}(V)(F))$ and has the same orbital integral as f_W .

Now note that $Z = \pi'(Y) \subseteq (\mathfrak{S}\!/H')(F)$ is compact. And D_μ is the preimage under $\tilde{\pi}$ of a hypersurface denoted by C in $(\mathfrak{S}\!/H')(F)$ such that $Z \cap C = \emptyset$. Then we may find a \mathcal{C}^∞ function β on $(\mathfrak{S}\!/H')(F)$ satisfying

1. $\beta|_Z \equiv 1$.
2. $\beta|_C \equiv 0$.

When F is archimedean, one may construct β using a bump function. When F is non-archimedean function, we may cover Z by open compact subsets \hat{Z} which is disjoint from C . Then we take β to be the characteristic function of \hat{Z} .

Then we may take α_W to be the pull-back of β under $\pi_{W,F}$.

The other direction can be proved similarly. \square

We have completed the proof of Theorem 3.3.

3.2 Reduction to local transfer around zero

The aim of this section is to reduce the existence of transfer to the existence of a local transfer near zero (Proposition 3.16). From now on we will denote by $Q_n = \mathbb{A}^{2n+1}$ or simply Q the common base $\mathfrak{S}_{n+1}\!/H' \simeq \mathfrak{U}(V)\!/H$ as an affine variety.

Localization. We fix a transfer factor ω and let $\pi' : \mathfrak{S}(F) \rightarrow Q(F)$ and $\pi : \mathfrak{U}(V)(F) \rightarrow Q(F)$ be the induced maps on the rational points.

Definition 3.7. Let Φ be a function on $Q(F)_{rs}$ which vanishes outside a compact set of $Q(F)$.

- (1) For $x \in Q(F)$ (not necessarily regular-semisimple), we say that Φ is a local orbital integral for \mathfrak{S} around $x \in Q(F)$ if there exists a neighborhood U of x in $Q(F)$ and a function $f \in \mathcal{C}_c^\infty(\mathfrak{S}(F))$ such that for all $y \in U_{rs}$, and z with $\pi'(z) = y$ we have

$$\Phi(y) = \omega(z)O(z, f).$$

- (2) Similarly we can define a local orbital integral for $\mathfrak{U}(V)$ around a point $x \in Q(F)$.

Note that if Φ is a local orbital integral for a transfer factor ω , it is also a local orbital integral for any other equivalent transfer factor $\omega' \sim \omega$.

We have the following localization principle for orbital integrals.

Proposition 3.8. Let Φ be a function on $Q(F)_{rs}$ which vanishes outside a compact set \mathcal{U} of $Q(F)$. If Φ is a local orbital integral for \mathfrak{S} at x for all $x \in Q(F)$, then it is an orbital integral, namely there exists $f \in \mathcal{C}_c^\infty(\mathfrak{S}(F))$ such that for all $y \in Q(F)_{rs}$, and z with $\pi'(z) = y$ we have

$$\Phi(y) = \omega(z)O(z, f).$$

Similar result holds for $\mathfrak{U}(V)$.

Proof. By assumption, for each $x \in \mathcal{U}$ we have an open neighborhood U_x and $f_x \in \mathcal{C}_c^\infty(\mathfrak{S}(F))$. By the compactness of \mathcal{U} , we may find finitely many of them, say x_1, \dots, x_m , such that U_{x_i} cover \mathcal{U} . Then we apply “partition of unity” to the cover of $Q(F)$ by U_{x_i} ($i = 1, 2, \dots, m$) and $Q(F) - \mathcal{U}$ to obtain smooth functions β_i, β on $Q(F)$ such that $\text{supp}(\beta_i) \subseteq U_{x_i}$ and $\text{supp}(\beta) \subseteq Q(F) - \mathcal{U}$ and $\beta + \sum_i \beta_i$ is the identify function on $Q(F)$. Since $\Phi\beta \equiv 0$, we may write $\Phi = \sum_{i=1}^m \Phi_i$ where $\Phi_i = \Phi\beta_i$ is a function on $Q(F)_{rs}$ which vanishes outside U_{x_i} . Then $\alpha_i = \beta_i \circ \pi'$ is a smooth $H'(F)$ -invariant function on $\mathfrak{S}(F)$ and $f_{x_i}\alpha_i \in \mathcal{C}_c^\infty(\mathfrak{S}(F))$. We claim that for every $y \in Q(F)_{rs}$ and $z \in \pi'^{-1}(y)$, we have $\omega(z)O(z, f_{x_i}\alpha_i) = \Phi_i(y)$. Indeed, the left hand side is equal to $\omega(z)O(z, f_{x_i})\beta(y)$. If y is outside U_{x_i} , then both sides vanish. If $y \in U_{x_i}$, then by the choice of f_{x_i} , we have $\omega(z)O(z, f_{x_i}) = \Phi(y)$. By the claim, we may take $f = \sum_i f_{x_i}\alpha_i$ to complete the proof. \square

For $f \in \mathcal{C}_c^\infty(\mathfrak{S}(F))$, we define a “direct image” $\pi'_{*,\omega}(f)$ as the function on $Q(F)_{rs}$:

$$\pi'_{*,\omega}(f)(x) := \omega(y)O(y, f),$$

where $x \in Q(F)_{rs}, y \in (\pi')^{-1}(x)$. It clearly does not depend on the choice of y . Similarly, for $f_W \in \mathcal{C}_c^\infty(\mathfrak{U}(V))$, we define a function $\pi_{W,*}(f_W)$ on $Q(F)_{rs}$ (extend by zero to those $x \in Q(F)_{rs}$ such that $\pi_W^{-1}(x)$ is empty). If the dependence of W is clear, we will also write it as $\pi_{*,\omega}(f_W)$ with the trivial transfer factor $\omega = 1$.

Definition 3.9. For $x \in Q(F)$ (not necessarily regular-semisimple), we say that the local transfer around x exists, if for all $f \in \mathcal{C}_c^\infty(\mathfrak{S}(F))$, there exist $(f_W)_W$ ($f_W \in \mathcal{C}_c^\infty(\mathfrak{U}(V))$) such that in a neighborhood of x in $Q(F)$, the following equality holds

$$\pi'_{*,\omega}(f) = \sum_W \pi_{W,*}(f_W),$$

and conversely for any tuple $(f_W)_W$, we may find f satisfying the equality.

Descent of orbital integrals We recall some results of [2]. Let \mathcal{V} be a representation of a reductive group H . Let $\pi : \mathcal{V} \rightarrow \mathcal{V}\!/H$ be the categorical quotient. An open subset $U \subset \mathcal{V}(F)$ is called *saturated* if it is the preimage of an open subset of $(\mathcal{V}\!/H)(F)$.

Let $x \in \mathcal{V}(F)$ be a semisimple element. Let $N_{Hx,x}^\mathcal{V}$ be the normal space of Hx at x . Then the stabilizer H_x acts naturally on the vector space $N_{Hx,x}^\mathcal{V}$. We call $(H_x, N_{Hx,x}^\mathcal{V})$ the *sliced representation at x* .

An étale Luna slice (for short, a Luna slice) at x is by definition ([2]) a locally closed smooth H_x -invariant subvariety $Z \ni x$ together with a strongly étale H_x -morphism $\iota : Z \rightarrow N_{Hx,x}^\mathcal{V}$ such that the H -morphism $\phi : H \times_{H_x} Z \rightarrow \mathcal{V}$ is strongly étale. Here, $H \times_{H_x} Z$ is $(H \times Z)\!/H_x$ for the action $h_x(h, z) = (hh_x^{-1}, h_x)$ and an H -morphism between two affine varieties $\phi : X \rightarrow Y$ is called *strongly étale* if $\phi\!/H : X\!/H \rightarrow Y\!/H$ is étale and the induced diagram is Cartesian:

$$\begin{array}{ccc} X & \xrightarrow{\phi} & Y \\ \downarrow & & \downarrow \\ X\!/H & \xrightarrow{\phi\!/H} & Y\!/H. \end{array}$$

When there is no confusion, we will simply say that Z is an étale Luna slice.

We then have the *Luna's étale slice theorem*: Let a reductive group H act on a smooth affine variety X and let $x \in X$ be semisimple. Then there exists a Luna slice at x . We will describe an explicit Luna slice in the appendix for our case. We may even assume that the morphism ι is essentially an open immersion in our case.

As an application, we have an analogue of Harish-Chandra's compactness lemma ([28, Lemma 25]).

Lemma 3.10. *Let $x \in \mathcal{V}(F)$ be semisimple. Let Z be an étale Luna slice at x . Then for any $H_x(F)$ -invariant neighborhood \mathcal{U} of x in $Z(F)$ whose image in $(Z//H_x)(F)$ is (relatively) compact, and any compact subset Ξ of $\mathcal{V}(F)$, the set*

$$\{h \in H_x(F) \backslash H(F) : \mathcal{U}^h \cap \Xi \neq \emptyset\}$$

is relatively compact in $H_x(F) \backslash H(F)$. Recall that the notation \mathcal{U}^h is given by (1.7).

Proof. We consider the étale Luna slice:

$$\phi : H \times_{H_x} Z \rightarrow \mathcal{V}.$$

Consider the composition:

$$H \times_{H_x} Z \simeq \mathcal{V} \times_{\mathcal{V}/H} Z//H_x \hookrightarrow \mathcal{V} \times Z//H_x.$$

The composition is a closed immersion. Shrinking Z if necessary, we may take the F -points to get a closed embedding

$$i : (H \times_{H_x} Z)(F) \hookrightarrow \mathcal{V}(F) \times (Z//H_x)(F).$$

We also have the projection

$$H \times_{H_x} Z = (H \times Z)//H_x \rightarrow H_x \backslash H.$$

We denote

$$j : (H \times_{H_x} Z)(F) \rightarrow (H_x \backslash H)(F).$$

Note that $H_x(F) \backslash H(F)$ sits inside $(H_x \backslash H)(F)$ as an open and closed subset. Let \mathcal{U}' be the image of \mathcal{U} in $(Z//H_x)(F)$. Then we see that the set

$$\{h \in H_x(F) \backslash H(F) : \mathcal{U}^h \cap \Xi \neq \emptyset\}$$

is contained in

$$ji^{-1}(\Xi \cap \mathcal{U}')$$

which is obviously compact. □

We also need the *analytic Luna slice theorem* ([2, Theorem 2.7]): there exists

- (1) an open $H(F)$ -invariant neighborhood U of $H(F)x$ in $\mathcal{V}(F)$ with an H -equivariant retract $p : U \rightarrow H(F)x$;
- (2) a H_x -equivariant embedding $\psi : p^{-1}(x) \hookrightarrow N_{H_x,x}^{\mathcal{V}}(F)$ with an open saturated image such that $\psi(x) = 0$.

$$\begin{array}{ccccc}
N_{H_x,x}^{\mathcal{V}}(F) & \xleftarrow{\psi} & p^{-1}(x) & \xrightarrow{\quad} & U \\
\uparrow & & \downarrow & & \downarrow p \\
0 & \xleftarrow{\quad} & x & \xrightarrow{\quad} & H(F)x
\end{array}$$

Denote $S = p^{-1}(x)$ and $N = N_{H_x,x}^{\mathcal{V}}(F)$. The quintuple (U, p, ψ, S, N) is then called an *analytic Luna slice* at x .

From an étale Luna slice we may construct an analytic Luna slice from (cf. the proof of [2, Corollary A.2.4]). In our case, the existence of analytic Luna slice is self-evident once we describe the explicit étale Luna slices in the appendix.

We recall some useful properties of analytic Luna slice ([2, Corollary 2.3.19]). Let $y \in p^{-1}(x)$, and $z := \psi(y)$. Then we have

- (1) $(H(F)_x)_z = H(F)_y$.
- (2) $N_{H(F)_y,y}^{\mathcal{V}(F)} = N_{H_x(F)_z,z}^N$ as $H(F)_y$ -spaces.
- (3) y is H -semisimple if and only if z is H_x -semisimple.

As an application, we state the Harish-Chandra (semisimple-) descent for orbital integrals.

Proposition 3.11. *Let $x \in \mathcal{V}(F)$ be semisimple and let (U, p, ψ, S, N) be an analytic Luna slice at x . Then there exists a neighborhood $\mathcal{U} \subset \psi(S)$ of 0 in $N_{H_x,x}^{\mathcal{V}}(F)$ with the following properties*

- To every $f \in \mathcal{C}_c^\infty(\mathcal{V}(F))$, we may associate $f_x \in \mathcal{C}_c^\infty(N_{H_x,x}^{\mathcal{V}}(F))$ such that for all semisimple $z \in \mathcal{U}$ (with $z = \psi(y)$) such that $\eta|_{H_y(F)} = 1$, we have

$$(3.8) \quad \int_{H_y(F) \backslash H(F)} f(y^h) \eta(h) dh = \int_{H_y(F) \backslash H_x(F)} f_x(z^h) \eta(h) dh.$$

- Conversely, given $f_x \in \mathcal{C}_c^\infty(N_{H_x,x}^{\mathcal{V}}(F))$, we may find $f \in \mathcal{C}_c^\infty(\mathcal{V}(F))$ such that (3.8) holds for all semisimple $z \in \mathcal{U}$ with $\eta|_{H_y(F)} = 1$.

Proof. Let \mathcal{U}' be a relatively compact neighborhood of x in S and let $\mathcal{U} = \psi(\mathcal{U}')$. By Lemma 3.10, we may find a compact set C of $H_x(F) \backslash H(F)$ that contains the set

$$\{h \in H_x(F) \backslash H(F) : \mathcal{U}'^h \cap \text{supp}(f) \neq \emptyset\}.$$

In the non-archimedean case, we may assume that C is compact open. Choose any function $\alpha \in \mathcal{C}_c^\infty(H(F))$ such that the function

$$H_x(F) \backslash H(F) \ni h \mapsto \int_{H_x(F)} \alpha(gh) dg$$

is the characteristic function 1_C in the non-archimedean case (or, in the archimedean case, a bump function that takes value one on C and zero outside some larger compact subset $C_1 \supset C$). We define a function on S by:

$$f_x(y) := \int_{H(F)} f(y^h) \alpha(h) \eta(h) dh.$$

In the non-archimedean case, we may assume that S is a closed subset of \mathcal{V} and in the archimedean case, we may assume that S contains a closed neighborhood of x in \mathcal{V} whose image in $N_{Hx,x}^{\mathcal{V}}(F)$ is the pre-image of a closed neighborhood in the categorical quotient. Then possibly using a bump function in the archimedean case to modify f_x , we may assume that $f_x \in \mathcal{C}_c^\infty(S)$. The map $f \mapsto f_x$ depends on \mathcal{U} . We may also view $f_x \in \mathcal{C}_c^\infty(N_{Hx,x}^{\mathcal{V}}(F))$ via the embedding $\psi : S \hookrightarrow N_{Hx,x}^{\mathcal{V}}(F)$.

Now the right hand side of (3.8) is equal to

$$\begin{aligned} & \int_{H_y(F) \setminus H_x(F)} \int_{H(F)} f(y^{hg}) \alpha(g) \eta(g) dg \eta(h) dh \\ &= \int_{H_y(F) \setminus H_x(F)} \int_{H(F)} f(y^g) \alpha(h^{-1}g) \eta(g) dg dh \\ &= \int_{H_y(F) \setminus H_x(F)} \int_{H_y(F) \setminus H(F)} \int_{H_y(F)} f(y^g) \alpha(h^{-1}pg) dp \eta(g) dg dh. \end{aligned}$$

Interchange the order of the first two integrals and notice that when $g \in C$

$$\int_{H_y(F) \setminus H_x(F)} \int_{H_y(F)} \alpha(h^{-1}pg) dp dh = \int_{H_x(F)} \alpha(h^{-1}g) dh = 1.$$

By Lemma 3.10, the value of the above integral outside C does not matter when $y \in \mathcal{U}'$. We thus obtain

$$\int_{H_y(F) \setminus H(F)} f(y^g) 1_C(g) \eta(g) dg.$$

This is equal to the left hand side when $y \in \mathcal{U}'$ (or equivalently, $\psi(y) = z \in \mathcal{U}$).

To show the converse, we note that $\psi(S)$ is saturated in $N_{Hx,x}^{\mathcal{V}}(F)$. Replacing f_x by $f_x \cdot 1_S$ in the non-archimedean case, and by $f_x \cdot \alpha_S$ for some bump function α in the archimedean case, we may assume that $\text{supp}(f_x) \subset \psi(S)$. Then we choose a function $\beta \in \mathcal{C}_c^\infty(H(F))$ such that

$$(3.9) \quad \int_{H(F)} \beta(h) \eta(h) dh = 1.$$

Consider the natural surjective map

$$H(F) \times S \rightarrow U$$

under which $H(F) \times S$ is a $H_x(F)$ -principal homogenous space over U (in the category of F -manifolds). It is obviously a submersion. We define $f \in \mathcal{C}_c^\infty(U)$ by integrating $\beta \otimes f_x$ over the fiber

$$f(y^h) := \int_{H_x(F)} f_x(\psi(y^g)) \beta(g^{-1}h) dg, \quad y \in S, h \in H(F).$$

Then $f \in \mathcal{C}_c^\infty(U)$ can be also viewed as an element in $\mathcal{C}_c^\infty(\mathcal{V}(F))$. The left hand side of (3.8) is then equal to

$$\begin{aligned} & \int_{H_y(F) \backslash H(F)} \int_{H_x(F)} f_x(\psi(y^g)) \beta(g^{-1}h) dg \eta(h) dh \\ &= \int_{H_y(F) \backslash H(F)} \int_{H_y(F) \backslash H_x(F)} \int_{H_y(F)} f_x(\psi(y^g)) \beta(g^{-1}p^{-1}h) dg \eta(h) dh \\ &= \int_{H_y(F) \backslash H_x(F)} \left(\int_{H(F)} \beta(g^{-1}h) \eta(h) dh \right) f_x(\psi(y^g)) dg. \end{aligned}$$

By (3.9), this is equal to

$$\int_{H_y(F) \backslash H_x(F)} f_x(\psi(y^g)) \eta(g) dg.$$

This completes the proof. \square

Smooth transfer for regular supported functions.

Lemma 3.12. *Let \mathcal{V} be either \mathfrak{S} or $\mathfrak{U}(V)$. Let $f \in \mathcal{C}_c^\infty(\mathcal{V}(F))$. Then the function $\pi_{*,\omega}(f)$ is smooth on $(\mathcal{V} \parallel H)(F)_{rs}$ and (relatively) compact supported on $(\mathcal{V} \parallel H)(F)$.*

Proof. The smoothness follows from the first part of Proposition 3.11 and the fact that the stabilizer of a regular semisimple element is trivial. The support is contained in the continuous image of a compact set, hence (relatively) compact. \square

Proposition 3.13. (1) *If $f' \in \mathcal{C}_c^\infty(\mathfrak{S}_{rs})$, then $\pi_{*,\omega} \in \mathcal{C}_c^\infty(Q(F)_{rs})$. Conversely, given $\phi \in \mathcal{C}_c^\infty(Q(F)_{rs})$ viewed as a function on $Q(F)$, there exists $f' \in \mathcal{C}_c^\infty(\mathfrak{S}_{rs})$ such that $\pi_{*,\omega}(f') = \phi$.*

(2) *If $f_W \in \mathcal{C}_c^\infty(\mathfrak{U}(V)_{rs})$, then $\pi_*(f_W) \in \mathcal{C}_c^\infty(Q(F)_{rs})$. Conversely, given $\phi \in \mathcal{C}_c^\infty(Q(F)_{rs})$ viewed as a function on $Q(F)$, there exists a tuple $(f_W \in \mathcal{C}_c^\infty(\mathfrak{U}(V)_{rs}))_W$ such that $\sum_W \pi_*(f_W) = \phi$.*

Proof. We only prove (1) and the proof of (2) is similar. By Lemma 3.12, it suffices to show the converse part. By the localization principle Proposition 3.8 (or rather its proof), it suffices to show that for every regular semisimple $x \in Q(F)$, ϕ is locally an orbital integral at x of a function with regular-semisimple support. We now fix a regular semisimple x . Note that the stabilizer of x is trivial. When choosing of the analytic slice, we may require that S is contained in the regular semisimple locus. Then the result follows from the decent of orbital integral, i.e., the second part of Proposition 3.11. \square

This immediately implies:

Theorem 3.14. *Given $f' \in \mathcal{C}_c^\infty(\mathfrak{S}_{rs})$, there exists its smooth transfer (f_W) such that $f_W \in \mathcal{C}_c^\infty(\mathfrak{U}(V)_{rs})$. Conversely, given a tuple $(f_W \in \mathcal{C}_c^\infty(\mathfrak{U}(V)_{rs}))_W$, there exists its smooth transfer $f' \in \mathcal{C}_c^\infty(\mathfrak{S}_{rs})$.*

In particular, this includes the existence of local transfer at a regular semisimple point $z \in Q(F)$.

We also emphasize that in Theorem 3.14, the local field F is allowed to be archimedean.

Reduction to local transfer around 0 of sliced representations. The result in the rest of this section relies on the results in the appendix B on the explicit construction of Luna slices. The construction is very technical and we decide to write it as an appendix. We need the explicit construction, instead of the abstract existence theorem, for at least one reason: we need to compare the transfer factors for the original and the sliced representations (Lemma 3.15 below).

We now fix $z \in Q(F)$. Within the fiber of z , there are one semisimple H' -orbit in \mathfrak{S}_{n+1} and finitely many semisimple H -orbits in $\mathfrak{U}(V)$. Note that there may be infinitely many non-semisimple orbits within the fiber. By the description of the sliced representations at semisimple elements in the appendix B, we know that they are products of lower dimensional vector spaces that are of the same type as \mathfrak{S} or $\mathfrak{U}(V)$ with possibly extending the base field F to a finite extension. So we may also speak of the local transfer around zero of those sliced representations.

To compare the local transfer at z , and at zero of the sliced representations, we need to compare their transfer factors. We may define an equivalent choice of transfer factors as follows. For $x = \begin{pmatrix} X & u \\ v & d \end{pmatrix} \in \mathfrak{gl}_{n+1, F}$ we define

$$\nu(x) = \det(u, Xu, X^2u, \dots, X^{n-1}u).$$

Then the transfer factor can be chosen as $\eta(\nu(x)) \in \{\pm 1\}$.

Lemma 3.15. *We may choose an H_x -invariant neighborhood of x such that for any y in this neighborhood, $\omega(y)$ is equal to a non-zero constant times $\omega(\psi(y))$.*

Proof. We only treat the two basic case: (1) $r = 0$, (2) $r = n$, where r is as in the appendix B. The general case can be reduced to those two by the same strategy as in the proof of Lemma B.4 in the appendix B. When $r = n$, namely x is regular semisimple so that H_x is trivial, the assertion follows since there is a neighborhood of x over which ω is a constant. When $r = 0$, using the notations in (B.3) we have

$$\nu(y) = \pm \nu(\psi(y)) \det(\text{ad}(Y), \mathfrak{gl}_{n+1}/\mathfrak{gl}_{n+1, Y_{11}})^{1/2},$$

where $\det(\text{ad}(Y), \mathfrak{gl}_{n+1}/\mathfrak{gl}_{n+1, Y_{11}})^{1/2}$ is a square root of $\det(\text{ad}(Y), \mathfrak{gl}_{n+1}/\mathfrak{gl}_{n+1, Y_{11}})$ (for example it can be given by the determinant of $\text{ad}(Y)$ on the upper triangular blocks). Since $\det(\text{ad}(X), \mathfrak{gl}_{n+1}/\mathfrak{gl}_{n+1, X_{11}}) \neq 0$, we may shrink the neighborhood if necessary such that $\eta(\nu(y))$ and $\eta(\nu(\psi(y)))$ differ only by a non-zero constant. \square

Proposition 3.16 (Reduction to zero). *Fix $z \in Q(F)$. If the local transfer around zero exists for the sliced representations at z , then the local transfer around z exists.*

Proof. By Proposition 3.11, the orbital integral of regular semisimple element near a semisimple element can be written as an orbital integral of regular semisimple elements near zero of the sliced representation. By our construction in the appendix B, the choice of étale and analytic Luna slices on \mathfrak{S}_{n+1} and $\mathfrak{U}(V)$'s can be made compatible. Moreover, the transfer factors are compatible with respect to the choice of analytic Luna slices by Lemma 3.15. This completes the proof. \square

Remark 11. We see that the reduction steps are along the same line as those in the classical endoscopic transfer by Langlands–Shelstad. The only non-trivial point is the explicit construction of the étale Luna slices which is slightly more involved than the case of adjoint action of a reductive group on its Lie algebra.

4 Smooth transfer for Lie algebra

In this section we prove Theorem 2.6. From now on we assume that F is a *non-archimedean* local field of characteristic zero.

4.1 A relative local trace formula

To simplify notations we unify the linear side and the unitary side in this subsection. Let F be a field and E be an étale F -algebra of rank two. Namely, E is either

- a quadratic field extension of F , or
- $F \times F$.

Let W be a free E -module of rank n and $(\cdot, \cdot) : W \times W \rightarrow E$ a non-degenerate Hermitian form. We denote by $H = H_W$ the algebraic group $U(W)$:

$$H = U(W) = \{h \in \text{Aut}_E(W) : (hu, hv) = (u, v), u, v \in W\}$$

and $\mathfrak{U}(W)$ its Lie algebra:

$$\mathfrak{U}(W) = \{X \in \text{End}_E(W) : (Au, v) = -(u, Av)\}.$$

Note that we allow $E = F \times F$, in which case we have $H \simeq \text{GL}_n$.

We will use $x \rightarrow \bar{x}$ to denote the (unique) non-trivial F -linear automorphism of E . In the case $E = F \times F$, it is the permutation of the two coordinates.

We consider the representation of $H = H_W$ on the F -vector space:

$$(4.1) \quad \mathcal{V} = \mathfrak{U}(W) \times W.$$

We usually denote by $x = (X, w)$ an element of \mathcal{V} for $X \in \mathfrak{U}(W)$, $w \in W$. We define

$$(4.2) \quad \Delta(x) = \Delta(X, w) = \det((X^i w, X^j w))_{i,j=0}^{n-1}.$$

Then x is regular semisimple if and only if $\Delta(x) \neq 0$. In the case $E = F \times F$, we also consider $\mathfrak{gl}_n \times F_n \times F^n$ with the natural action of $H = \mathrm{GL}_n$ by

$$h \cdot (X, u, v) = (h^{-1}Xh, uh^{-1}, hv).$$

Similarly we define for an element $x = (X, u, v)$

$$(4.3) \quad \Delta(x) = \Delta(X, u, v) = \det(uX^{i+j}v)_{i,j=0}^{n-1}.$$

Then x is regular semisimple if and only if $\Delta(x) \neq 0$. In either case, we denote by $D(X)$ the discriminant of $X \in \mathfrak{U}(W)$ or \mathfrak{gl}_n , namely

$$(4.4) \quad D(X) = \prod_{i,j} (\lambda_i - \lambda_j)^2,$$

where $\lambda_1, \lambda_2, \dots, \lambda_n$ are the n eigenvalues of X .

Upper bound of orbital integrals. We first estimate the orbital integral for a regular semisimple $x \in \mathcal{V}$:

$$(4.5) \quad O_x(f) = O(x, f) := \int_H f(x^h)\eta(h)dh, \quad f \in \mathcal{C}_c^\infty(\mathcal{V}),$$

where η is any (unitary) character. Let \mathfrak{h} be the Lie algebra of $H = H_W$. Then we have $\mathcal{V} = \mathfrak{h} \times W$.

Lemma 4.1. *Let $\Omega \subset \mathfrak{h}(F)$ be a compact open set. Let T be a Cartan subgroup of $H(F)$ and $\mathfrak{t} \subset \mathfrak{h}(F)$ be the corresponding Cartan subalgebra. Let $\varphi \in \mathcal{C}_c^\infty(W)$. Then there exists a constant $r > 0$ depending only on n and a constant C depending only on n, φ, Ω with the following property: for all regular semisimple $X \in \mathfrak{t}$, and $h \in \mathrm{GL}_n(F)$ such that $X^h \in \Omega$, we have*

$$\left| \int_T \varphi(w^h)dt \right| \leq C \cdot \max\{1, \log|\Delta(X, w)|\}^r.$$

Proof. We prove this in the general linear case. The unitary case is similar and easier. We write $\mathcal{V} = \mathfrak{gl}_n \times F_n \times F^n$ and $x = (X, u, v)$. If $h^{-1}Xh \in \Omega$, then for all $i, j = 0, \dots, n-1$, the following vectors are in a compact set depending only on the support of φ and Ω :

$$h^{-1}t^{-1}X^i v, \quad uX^j t h.$$

Write $\delta_1 = (X^i v)_{i=0, \dots, n-1} \in \mathfrak{gl}_n(F)$ and $\delta_2 = (uX^j)_{j=0, \dots, n-1} \in \mathfrak{gl}_n(F)$ so that $\Delta(X, u, v) = \det(\delta_1 \delta_2)$. Then the condition becomes that the elements

$$h^{-1}t^{-1}\delta_1, \quad \delta_2 t h,$$

are in a compact set Ω_1 of $\mathfrak{gl}_n(F)$ depending only on the support of φ and Ω .

We may identify \mathfrak{t} with $\prod_{i=1}^e E_i$ and T with $\prod_{i=1}^e E_i^\times$ where E_i/F is a degree n_i field extension such that $\sum n_i = n$. Let P be the parabolic of $H = \mathrm{GL}_{n-1}$ associated to the

partition $\sum n_i = n$ with Levi decomposition $P = MN$ and we may assume that $T \subset M$. Then T is elliptic in M . By Iwasawa decomposition $H = NMK$ for $K = \mathrm{GL}_n(\mathcal{O}_F)$, we may write $h = nmk$, $\delta_1 = n_1 m_1 k_1$, $\delta_2 = k_2 m_2 n_2$. By enlarging Ω_1 if necessary, we may assume that $\Omega_1 = K\Omega_1 K$ (i.e., bi- K -invariant). Since $h^{-1}t\delta_1 = k^{-1}m^{-1}n^{-1}tn_1 m_1 k_1 \in \Omega_1$, we may see that the Levi component $m^{-1}t^{-1}m_1 \in \Omega_2$ for a compact set Ω_2 of $M_2(F)$. Similarly we have $m_2 t m \in \Omega_2$. Let $t = (t^{(i)})_{i=1}^e \in T$ where $t^{(i)} \in E_i^\times$, and similarly for m, m_1, m_2 . Then there exist constants $C_1, C_2 > 0$ such that for all $i = 1, 2, \dots, e$:

$$C_1 |\det(m_1^{(i)}(m^{(i)})^{-1})| \leq |t_i| \leq C_2 |\det(m_2^{(i)} m^{(i)})|^{-1}.$$

The volume of such t_i is bounded above by

$$C_3 + \log \frac{C_2 |\det(m_2^{(i)} m^{(i)})|^{-1}}{C_1 |\det(m_1^{(i)}(m^{(i)})^{-1})|} = C_4 - \log |\det(m_1^{(i)} m_2^{(i)})|$$

for constants C_3, C_4 . In particular, the integral in the lemma is zero unless for all $i = 1, 2, \dots, e$ we have

$$C_4 - \log |\det(m_1^{(i)} m_2^{(i)})| \geq 0.$$

Under this assumption, there is a constant C such that for all $i = 1, 2, \dots, e$:

$$C_4 - \log |\det(m_1^{(i)} m_2^{(i)})| \leq C_5 - \sum_{i=1}^e \log |\det(m_1^{(i)} m_2^{(i)})|,$$

which is equal to

$$C_5 - \log |\det(m_1 m_2)| = C_5 - \log |\det(\Delta(X, u, v))|.$$

Then it is easy to see that the integral over T is either zero or bounded above by the L^∞ -norm of φ times

$$(C_5 - \log |\det(\Delta(X, u, v))|)^k.$$

Now the lemma follows immediately. \square

Before we proceed, we introduce one more definition: we say that $x = (X, w)$ is *strongly regular* if x, X, w are all H -regular semisimple (ie., $\Delta(x) \neq 0$, $(w, w) \neq 0$ and $D(X) \neq 0$).

Proposition 4.2. *Let $f \in \mathcal{C}_c^\infty(\mathcal{V})$. Then there exist a constant C depending on f , and an integer $r > 0$, such that for all $x \in \mathcal{V}$ strongly regular:*

$$|O(x, f)| \leq C \cdot \max\{1, |\log |\Delta(x)||^r\} \max\{1, |D(X)|^{-1/2}\}.$$

Proof. We only give the proof in the general linear case. The unitary case is similar and easier. We choose a (finite) complete set of representatives of Cartan subalgebras \mathfrak{t} up to H -conjugacy. We may assume that $X \in \mathfrak{t}$ and $f = \phi \otimes \varphi$ where $\phi \in \mathcal{C}_c^\infty(\mathfrak{gl}_n(F))$ and $\varphi \in \mathcal{C}_c^\infty(F_n \times F^n)$ as in Lemma 4.1. Then we have Now we have

$$|O(x, f)| \leq C_6 \cdot \max\{1, |\log |\Delta(x)||\} \int_{H/T} |\phi(ht_0 h^{-1})| dh.$$

By the bound of Harish-Chandra ([29]) on the usual orbital integral, the integral in the right hand side is bounded by a constant times $\max\{1, |D(X)|^{-1/2}\}$. Since there are only finitely many \mathfrak{t} , we may choose a uniform constant C to complete the proof. \square

Local integrability. We want to show the orbital integral, as a function on \mathcal{V} , is locally integrable. The following result is probably well-known. But we could not find a reference so we decide to include a proof here.

Lemma 4.3 (Igusa integral). *Let $P(x) \in F[x_1, \dots, x_m]$ be a polynomial. Then there exists $\epsilon > 0$ such that*

$$\int_{\mathcal{O}_F^m} |P(x)|^{-\epsilon} dx < \infty.$$

If P is homogeneous, then there exists $\epsilon > 0$ such that the function $|P(x)|^{-\epsilon}$ is locally integrable everywhere on F^m .

Proof. The first assertion implies the second one. If P is homogenous of degree $k \geq 0$, assuming the first assertion we want to show that $|P|^{-\epsilon}$ is locally integrable around any $x_0 \in F^m$. Indeed, we may assume that $x_0 \in \varpi^{-n}\mathcal{O}^m$ for some $n > 0$. By homogeneity, we have

$$\int_{\varpi^{-n}\mathcal{O}^m} |P(x)|^{-\epsilon} dx = |\varpi|^{-mn+\epsilon kn} \int_{\mathcal{O}^m} |P(x)|^{-\epsilon} dx < \infty.$$

This shows the local integrality around x_0 .

To show the first assertion, we may assume that $P \in \mathcal{O}_F[x]$ and $F = \mathbb{Q}_p$. Now following Igusa ([9]) we define

$$\tilde{N}_n := \#\{x \in (\mathbb{Z}_p/p^n)^m \mid f(x) \equiv 0 \pmod{p^n}\}.$$

Let $w_n = \tilde{N}_n/p^{nm}$. Then $w_{n+1} < w_n$ and we want to prove that there exists $\epsilon > 0$ such that

$$\sum_n p^{n\epsilon}(w_n - w_{n+1}) < \infty.$$

We define an associated Poincare series

$$\tilde{P}(T) = \sum_{n=0}^{\infty} \tilde{N}_n T^n.$$

By the rationality of $\tilde{P}(T)$ (proved first by Igusa, [9]), we may write

$$\tilde{P}(T) = Q(T) \prod_{i,j} (1 - \alpha_{i,j} p^{\beta_i} T)^{-k_{i,j}},$$

where $\beta_i \in \mathbb{R}$, $k_i \in \mathbb{N}_{>0}$, and β_i are distinct, $\alpha_{i,j}$ are roots of unity, $Q(T)$ is a polynomial ([9, Remark 3.3]). We must have all $\beta_i \leq m$ since $\tilde{N}_n \leq |(\mathbb{Z}_p/p^n)^m| = q^{nm}$ for all n . If all $\beta_i < m$, then we certainly can choose $\epsilon > 0$ such that all $\beta_i < m - \epsilon$ and hence

$$\tilde{N}_n = O(p^{n(m-\epsilon)}).$$

Assume now that $\beta_0 = m$ and all other $\beta_i < m$. Since $|\tilde{N}_n| \leq p^{nm}$ we must have all $k_{0,j} = 1$ (i.e., no multiplicity). Then for suitable $a_j \in \mathbb{C}$, and all $\epsilon < m - \max_{i \neq 0} \{\beta_i\}$, we have, when n is sufficiently large

$$|\tilde{N}_n - p^{nm} \sum_j a_j \alpha_{0,j}^n| = O(p^{n(m-\epsilon)}).$$

Let $w'_n = \sum_j a_j \alpha_{0,j}^n$. Since $\alpha_{i,j}$ are roots of unity, the sequence w'_n ($n \geq 0$) is periodic, say with period N , i.e.: $w'_n = w'_{n+N}$. Then by

$$|w_n - w'_n| = O(p^{-n\epsilon}),$$

we have

$$w_n - w_{n+1} \leq w_n - w_{n+N} \leq |w'_n - w'_{n+N}| + O(p^{-n\epsilon}) = O(p^{-n\epsilon}).$$

This completes the proof. \square

Remark 12. The integral in the lemma is called *local zeta integral*. The same result also holds for archimedean local fields.

Corollary 4.4. *There exists $\epsilon > 0$ such that the function*

$$m_\epsilon : x = (X, w) \in \mathcal{V} \mapsto |D(X)|^{-1/2-\epsilon} \log |\Delta(x)|$$

is locally integrable on \mathcal{V} .

Proof. Let Ω be a compact subset of \mathcal{V} . In Young's inequality

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \quad a, b, p, q > 0, \frac{1}{p} + \frac{1}{q} = 1,$$

we let $p = 1 + \epsilon_1$ to obtain

$$m_\epsilon(x) \leq \frac{|D(X)|^{-(1/2+\epsilon)(1+\epsilon_1)}}{1 + \epsilon_1} + \frac{|\log |\Delta(x)||^q}{q}.$$

We now need to use the Lie algebraic version of [28, Theorem 15], namely: there exists $\epsilon_2 > 0$ such that the function $X \mapsto |\Delta(X)|^{-1/2-\epsilon_2}$ is locally integrable on $\mathfrak{U}(W)$. This implies that for an appropriate choice of ϵ, ϵ_1 , the first term above is locally integrable on $\mathfrak{U}(W) \times W$. Lemma 4.3 implies that the second term is also locally integrable. This completes the proof. \square

In summary we have showed that

Corollary 4.5. *For any $f \in \mathcal{C}_c^\infty(\mathcal{V})$, we have*

- *The absolute value of the orbital integral $x \mapsto |O(x, f)|$ is locally integrable on \mathcal{V} .*
- *If $X \in \mathfrak{h}(F)$ is regular semisimple, the function $w \mapsto |O((X, w), f)|$ is locally integrable on W .*
- *If $w \in W$ is regular semisimple, the function $X \mapsto |O((X, w), f)|$ is locally integrable on $\mathfrak{h}(F)$.*

A relative local trace formula We now show a local trace formula for the H -action on \mathcal{V} . We fix an H -invariant bilinear form

$$\langle \cdot, \cdot \rangle : \mathcal{V} \times \mathcal{V} \rightarrow F$$

such that its restriction to any invariant subspace is non-degenerate (obviously such form exists). We would like to consider the partial Fourier transform with respect to an invariant subspace \mathcal{V}_0 of \mathcal{V} . Let \mathcal{V}_0^\perp be the orthogonal complement of \mathcal{V}_0 and we write $x = (y, z)$ according to the decomposition $\mathcal{V} = \mathcal{V}_0 \oplus \mathcal{V}_0^\perp$. We define a partial Fourier transform $\mathcal{F}_{\mathcal{V}_0} f$:

$$(4.6) \quad \mathcal{F}_{\mathcal{V}_0} f(y, z) = \int_{\mathcal{V}_0} f(y', z) \psi(\langle y, y' \rangle) dy', \quad f \in \mathcal{C}_c^\infty(\mathcal{V}), \quad y \in \mathcal{V}_0, z \in \mathcal{V}_0^\perp.$$

We will choose the self-dual measure on \mathcal{V}_0 . Then the fact that Fourier transform is an isometry on L^2 space can be written as:

$$(4.7) \quad \int_{\mathcal{V}_0} f_1(y) \bar{f}_2(y) dy = \int_{\mathcal{V}_0} \mathcal{F}_{\mathcal{V}_0} f_1(y) \overline{\mathcal{F}_{\mathcal{V}_0} f_2(y)} dy.$$

It is clear that, for two orthogonal subspace $\mathcal{V}_0, \mathcal{V}_1$:

$$\mathcal{F}_{\mathcal{V}_0 \oplus \mathcal{V}_1} = \mathcal{F}_{\mathcal{V}_0} \circ \mathcal{F}_{\mathcal{V}_1} = \mathcal{F}_{\mathcal{V}_1} \circ \mathcal{F}_{\mathcal{V}_0}.$$

Returning to our case, \mathcal{V} is $\mathfrak{U}(W) \times W$ (i.e., either $\mathfrak{gl}_n \times F_n \times F^n$ for $E = F \times F$, or $\mathfrak{U}(W) \times W$ for a Hermitian space W for a quadratic field E extension). In each case we have an abelian 2-group of Fourier transforms generated by the two partial transforms $\mathcal{F}_{\mathfrak{U}(W)}, \mathcal{F}_W$.

We now ready to prove a *local relative trace formula for Lie algebras*. The name comes from the analogous (but much more difficult to prove) result of Waldspurger ([57]).

Theorem 4.6. *Let \mathcal{V} be $\mathfrak{U}(W) \times W$ and \mathcal{V}_0 an invariant-subspace. We write $x = (y, z)$ according to the decomposition $\mathcal{V} = \mathcal{V}_0 \oplus \mathcal{V}_0^\perp$. Fix a regular semisimple $z \in \mathcal{V}_0^\perp$. For $f_1, f_2 \in \mathcal{C}_c^\infty(\mathcal{V})$, we define an iterated integral*

$$(4.8) \quad T(f_1, f_2) = \int_{\mathcal{V}_0} \left(\int_{H(F)} f_1((y, z)^h) \eta(h) dh \right) f_2(y, z) dy,$$

where η is trivial in the unitary case. Then we have

$$(4.9) \quad T(f_1, f_2) = T(\hat{f}_1, \check{f}_2),$$

where \hat{f} (\check{f} , resp.) denotes the partial Fourier transform associated to \mathcal{V}_0 , with respect to ψ (ψ_{-1} , the complex conjugate of ψ , resp.).

Proof. We consider the unitary case. The linear case is similar. The idea is the same as in Harish-Chandra's work on the representability of the character of a supercuspidal representation. Take a sequence of increasing compact subsets Ω_i of H such that $H = \cup_{i=1}^\infty \Omega_i$. Define

$$O_i(x, f) = \int_{\Omega_i} f(x^h) dh.$$

Then it is clear that, for a regular semisimple X :

$$O(x, f) = \lim_{i \rightarrow \infty} O_i(x, f).$$

By (4.7) we have for any $f_1, f_2 \in \mathcal{C}_c^\infty(\mathcal{V})$:

$$\int_{\mathcal{V}_0} f_1((y, z)^h) f_2(y, z) dy = \int_{\mathcal{V}_0} \hat{f}_1((y, z)^h) \check{f}_2(y, z) dy,$$

noting that the Fourier transform commutes with the H -action. Therefore we have

$$(4.10) \quad \lim_{i \rightarrow \infty} \int_{\Omega_i} \left(\int_{\mathcal{V}_0} f_1((y, z)^h) f_2(y, z) dy \right) dh = \lim_{i \rightarrow \infty} \int_{\Omega_i} \left(\int_{\mathcal{V}_0} \hat{f}_1((y, z)^h) \check{f}_2(y, z) dy \right) dh.$$

As Ω_i is compact, we may interchange the order of integration. Obviously we have $|O_i((y, z), f_1)| \leq |O_i((y, z), |f_1|)|$. When z is regular semisimple, by Corollary 4.5 the function $y \mapsto |O((y, z), |f_1|)|$ is locally integrable on \mathcal{V}_0 . By Lebesgue's dominated convergence theorem we obtain

$$\lim_{i \rightarrow \infty} \int_{\Omega_i} \left(\int_{\mathcal{V}_0} f_1((y, z)^h) f_2(y, z) dy \right) dh = \int_{\mathcal{V}_0} O((y, z), f_1) f_2(y, z) dy = T(f_1, f_2).$$

Similarly we obtain $T(\hat{f}_1, \check{f}_2)$ from the right hand side of (4.10). This completes the proof. \square

A consequence. Now we specialize to the case $n = 1$. We deduce the representability of the Fourier transform of an orbital integral on $W = M \times M^*$ where M is a one dimensional F -vector space, M^* its dual. We then have $H = \mathrm{GL}(M) \simeq \mathrm{GL}_{1,F}$ acting on W .

Corollary 4.7. *For any quadratic character η (possibly trivial), the Fourier transform of the orbital integral*

$$(4.11) \quad \hat{O}_w(f) := \omega(w) \int_H \hat{f}(w^h) \eta(h) dh, \quad f \in \mathcal{C}_c^\infty(W),$$

(here $\omega(w)$ is the transfer factor defined in (3.7) for $n = 1$) is represented by a locally constant H -invariant function on the regular semisimple locus W_{rs} denoted by $\kappa^\eta(w, \cdot)$, i.e.: for any $f \in \mathcal{C}_c^\infty(W)$ we have

$$\hat{O}_w(f) = \int_W f(w') \omega(w') \kappa^\eta(w, w') dw'.$$

The same result holds for the unitary case. Moreover, we may let F be a product of fields and the same result holds.

Proof. The proof is along the same line as the proof of Harish-Chandra of the representability of Fourier transform of orbital integrals ([29, Lemma 1.19, pp.12]). With the local trace formula Theorem 4.6, it remains to note that the Howe's finiteness conjecture holds for the H -action on W , by [49, Theorem 6.1]. In the next two subsections we will prove a more explicit result when η is nontrivial quadratic. \square

Obviously the kernel function $\kappa^\eta(w, w')$ can be viewed as a locally constant function on $W_{rs} \times W_{rs}$ invariant under $H \times H$.

Remark 13. Unfortunately, the Howe’s finiteness conjecture (cf. [49, Theorem 6.1]) fails for the H -action on $\mathfrak{gl}_n \times F_n \times F^n$ when $n \geq 2$. Therefore the proof of Harish-Chandra ([29]) does not work and the representability of the Fourier transform of H -orbital integral remains open when $n \geq 2$.

4.2 A Davenport–Hasse type relation

We show a Davenport–Hasse type relation between two “Kloosterman sums”. It will be used to show that the Fourier transforms perserve smooth transfer when $n = 1$.

Local constants. We first recall the definition and some basic properties of the Langlands constant ([4, §29, §30]). Fix a non-trivial character ψ of F . For a field extension K/F of local fields, we define a character of K by $\psi_K = \psi \circ \text{tr}_{K/F}$. Let 1_K be the trivial character of the Weil group W_K . Then the Langlands constant is defined to be

$$(4.12) \quad \lambda_{K/F}(\psi) := \frac{\epsilon(\text{Ind}_{K/F} 1_K, s, \psi)}{\epsilon(1_K, s, \psi_K)},$$

which is a constant independent of s ([4, §30]). In particular, it is given by

$$(4.13) \quad \lambda_{K/F}(\psi) := \epsilon(\text{Ind}_{K/F} 1_K, 1/2, \psi).$$

The character of the abelianization $W_F^{ab} \simeq F^\times$:

$$(4.14) \quad \eta_{K/F} := \det(\text{Ind}_{K/F} 1_K)$$

is a quadratic character. For $a \in F^\times$ we denote by ψ_a the character of F defined by $\psi_a(x) = \psi(ax)$. We then have

$$\lambda_{K/F}(\psi_a) = \eta_{K/F}(a) \lambda_{K/F}(\psi).$$

Moreover we have

$$\lambda_{K/F}(\psi)^2 = \eta_{K/F}(-1).$$

In particular, $\lambda_{K/F}(\psi) \in \mu_4$ is a fourth root of unity.

We first show that the epsilon factor is essentially a Gauss sum. Let ψ be a nontrivial additive character and dx be the self-dual measure on F . We choose a Haar measure $d^\times x = \frac{dx}{|x|}$ on F^\times . For a quasi-character χ of F^\times , we may define its real exponent $\text{Re}(\chi)$ to be the unique real number r such that $|\chi(x)| = |x|^r$ for all $x \in F^\times$. We denote $\hat{\chi} = \chi^{-1}|\cdot|$. Let $\gamma(\chi, \psi)$ be the gamma factor in Tate’s thesis ([4, §23]).

Lemma 4.8. *The gamma factor $\gamma(\chi, \psi)$ as a meromorphic function of χ (namely, its value at $\chi|\cdot|^s$ is meromorphic for $s \in \mathbb{C}$) is given by*

$$(4.15) \quad \gamma(\chi, \psi) = \int_{F^\times} \psi(x) \hat{\chi}(x) \frac{dx}{|x|} = \int_F \psi(x) \chi^{-1}(x) dx.$$

Here the right hand side is interpreted as

$$\int_{|x|<C} \psi(x)\widehat{\chi}(x)|x|^s \frac{dx}{|x|} \Big|_{s=0}$$

for any C large enough. In particular, the gamma factor $\gamma(\chi, \psi)$ is holomorphic when $\operatorname{Re}(\chi) < 1$ and its value is given by an absolutely convergent integral

$$\gamma(\chi, \psi) = \int_{|x|<C} \psi(x)\chi^{-1}(x)dx$$

when C large enough.

Proof. By definition, we have ([4, §23])

$$\gamma(\chi, \psi) = \frac{\zeta(\widehat{f}, \widehat{\chi})}{\zeta(f, \chi)}.$$

Note that it does not depend on the choice of any Haar measure on F^\times . We have chosen $\frac{dx}{|x|}$. Let $f_n = 1_{1+\varpi^n\mathcal{O}_F}$. Then we have

$$\widehat{f}_n(x) = \psi(x)\widehat{1}_{\varpi^n\mathcal{O}_F}(x) = \psi(x)|\varpi|^n\widehat{1}_{\mathcal{O}_F}(x\varpi^n).$$

If n is larger than the conductor of χ , we have

$$\zeta(f_n, \chi) = |\varpi|^n \operatorname{vol}(\mathcal{O}_F).$$

Let m be the integer such that $\widehat{1}_{\mathcal{O}_F} = \operatorname{vol}(\mathcal{O}_F) \cdot 1_{\varpi^{-m}\mathcal{O}_F}$. Then we have

$$\begin{aligned} \zeta(\widehat{f}_n, \widehat{\chi}) &= \int_{F^\times} \psi(x)|\varpi|^n\widehat{1}_{\mathcal{O}_F}(x\varpi^n)\widehat{\chi}(x) \frac{dx}{|x|} \\ &= \operatorname{vol}(\mathcal{O}_F)|\varpi|^n \int_{\varpi^{-n-m}\mathcal{O}_F} \psi(x)\widehat{\chi}(x) \frac{dx}{|x|}. \end{aligned}$$

The right hand side is interpreted in the sense of analytic continuation. Therefore we have for n large enough:

$$\gamma(\chi, \psi) = \int_{\varpi^{-n-m}\mathcal{O}_F} \psi(x)\widehat{\chi}(x) \frac{dx}{|x|}.$$

This completes the proof. □

Kloosterman sums. We have some Kloosterman sums. Let E/F be a (genuine) quadratic field extension. Our first Kloosterman sum is defined for $a = A\bar{A} \in \operatorname{NE}^\times$:

$$(4.16) \quad \Phi(a) = \Phi_{E/F}(a) := \int_{E^1} \psi_E(Ax)d^\times x.$$

The measure on E^1 is chosen such that for all $\phi \in \mathcal{C}_c^\infty(E)$:

$$\int_{E^\times} \phi(X) \frac{dX}{|X|_E} = \int_{F^\times} \int_{E^1} \phi(At) dt d^\times a,$$

where $NA = a$ and dX is the self-dual measure on E with respect to the character ψ_E .

Our second Kloosterman sum is defined for $a \in F^\times$:

$$(4.17) \quad \Psi(a) = \Psi_{E/F}(a) := \int_{F^\times} \psi\left(x + \frac{a}{x}\right) \eta(x) d^\times x, \quad \eta = \eta_{E/F}.$$

The right hand side is interpreted as

$$(4.18) \quad \int_{1/C < |x| < C} \psi\left(x + \frac{a}{x}\right) \eta(x) d^\times x$$

for sufficiently large C (depending on a). Note that this integral becomes a constant when C is large enough.

We also define some auxiliary functions on F^\times indexed by $C \in \mathbb{R}$:

$$(4.19) \quad \Psi_C(a) := \int_{|a|/C < |y| < C} \psi(a/y + y) \eta(y) d^\times y, \quad a \in F^\times.$$

We set $\Psi_\infty = \Psi$. It is clear that, as $C \rightarrow \infty$, the functions Ψ_C converge to Ψ_∞ pointwisely.

Lemma 4.9. *The function Ψ_C (possibly $C = \infty$) is a locally constant function on F^\times . Moreover, there are constants β_1, β_2 independent of $C \in \mathbb{R} \cup \{\infty\}$ such that*

$$|\Psi_C(a)| \leq \beta_1 |\log |a|| + \beta_2$$

for all $a \in F^\times$.

Proof. Let B be such that $\psi(x) = 1$ if and only if $|x| \leq B$. If $|a| < B^2$, either $|x| < B$ or $|a/x| < B$. So we may bound the integral (4.18) by

$$\left| \int_{|x| < B, |a|/C < |x| < C} \psi(a/x) \eta(x) d^\times x \right| + \left| \int_{|x| \geq B, |a|/C < |x| < C} \psi(x) \eta(x) d^\times x \right|.$$

Since ψ is oscillating when $|x| \geq B$, it is easy to see that the second term is bounded above by a constant independent of a and C . The first term is equal to

$$\begin{aligned} & \left| \int_{|x| > a/B, |a|/C < |x| < C} \psi(x) \eta(x) d^\times x \right| \\ & \leq \left| \int_{a/B < |x| < B, |a|/C < |x| < C} \psi(x) \eta(x) d^\times x \right| + \left| \int_{|x| \geq B, |a|/C < |x| < C} \psi(x) \eta(x) d^\times x \right|. \end{aligned}$$

The first term of the last line is at most

$$\int_{a/B < |x| < B} 1 d^\times x,$$

which is of the form $\beta_1 |\log |a|| + \beta_2$ for some constants β_1, β_2 . Now, possibly enlarging β_2 by a constant independent of a and C , we complete the proof. \square

We now study the asymptotic behavior of $\Psi_C(a)$ and $\Phi(a)$ when $|a|$ is large.

Lemma 4.10. *There is constants A and α independent of $C \in \mathbb{R} \cup \{\infty\}$ such that when $|a| > A$, we have*

$$|\Psi_C(a)| < \alpha|a|^{-1/4}, \quad |\Phi(a)| < \alpha|a|^{-1/4}.$$

Proof. The proof follows the strategy of [57, Proposition VIII.1]. We only prove the case for Ψ_C since the same proof with simple modification also applies to Φ .

Denote by $v : F^\times \rightarrow \mathbb{Z}$ the valuation on F . We may choose a constant c such that whenever $m \geq c$, the exponential map

$$\begin{aligned} \varpi^m \mathcal{O}_F &\rightarrow F^\times \\ t &\mapsto e^t = \sum_{i \geq 0} \frac{t^i}{i!} \end{aligned}$$

converges and we have for $t \in \varpi^m \mathcal{O}_F$:

$$(*) \quad \begin{cases} v\left(\frac{e^t + e^{-t}}{2} - 1\right) = v(t^2/2). \\ v\left(\frac{e^t - e^{-t}}{2}\right) = v(t). \end{cases}$$

Let K_m be the image of $\varpi^m \mathcal{O}_F$. It is easy to see that $K_m = 1 + \varpi^m \mathcal{O}_F$. Let d be the conductor of ψ , i.e.: $\psi(x) = 1$ if $v(x) \leq d$ and $\psi(x) \neq 1$ for some $x \in F$ with $v(x) = d - 1$.

Now we choose $\ell \in \mathbb{Z}$ such that

$$(**) \quad \ell > 4c - 2d + 10.$$

Now assume that $v(a) < -\ell$. To explain the idea, we first claim that *when $v(a) < -\ell$, $\Psi_C(a) = 0$ unless $a \in (F^\times)^2$* . To see this, suppose that a is not a square. Then we have $|a/x \pm x| = \max\{|x|, |a/x|\} \geq |a|^{1/2}$ and

$$(4.20) \quad \Psi_C(a) = \sum_{v(a/C) < i < v(C)} \int_{v(x)=i} \psi(ax^{-1} + x)\eta(x)d^\times x.$$

Where we understand $v(C) = -\log_q C$, where $q = \#\mathcal{O}_F/(\varpi)$. For a fixed i , let $n > c$ be such that

$$n + \min\{i, v(a) - i\} < d < 2n + \min\{i, v(a) - i\}.$$

Such n exists due to (**). Then we have a nontrivial character of $\varpi^n \mathcal{O}_F$ defined by:

$$\begin{aligned} t &\mapsto \psi(ax^{-1}e^{-t} + xe^t)\eta(x) \\ &= \psi((ax^{-1} - x)t + (ax^{-1} + x)t^2/2 + \dots)\eta(x) \\ &= \psi((ax^{-1} - x)t)\eta(x). \end{aligned}$$

Hence in (4.20) the integral over $v(x) = i$ can be broken into a sum of integrals over xK_n where x runs over $\varpi^i \mathcal{O}_F / K_n$. Using the exponential map, each term is of the form:

$$\begin{aligned} & |x|^{-1} \int_{K_n} \psi(ax^{-1}k^{-1} + xk)\eta(x)d^\times k \\ &= |x|^{-1} \int_{\varpi^n \mathcal{O}_F} \psi(ax^{-1}e^{-t} + xe^t)\eta(x)dt \\ &= |x|^{-1} \eta(x) \psi((ax^{-1} - x)t) dt = 0. \end{aligned}$$

This completes the proof the claim.

Now we may assume that $a = b^2$ is a square. A change of variable yields:

$$(4.21) \quad \Psi_C(a) = \eta(b) \int_{|b/C| < x < |C/b|} \psi(b(x^{-1} + x))\eta(x)d^\times x.$$

For a fixed x , we look for an integer n such that

$$\begin{cases} n + v(b) + v(x - x^{-1}) < d, \\ 2n + v(b) > d. \end{cases}$$

For example we may take $n = 1 + [(d - v(b))/2] > c$ (due to (**)). Then the condition becomes:

$$(***) \quad v(x - x^{-1}) < d - 1 - v(b) - [(d - v(b))/2] = [(d - 1 - v(b))/2].$$

If this last inequality holds, we have $v(xk - x^{-1}k^{-1}) = v(x - x^{-1})$ for $k \in K_n$: this is obvious if $v(x) \neq 0$; if $v(x) = 0$, then the difference

$$(x - x^{-1}) - (xk - x^{-1}k^{-1}) = x(1 - k)(1 + x^{-2}k^{-1})$$

has valuation at least $v(1 - k) \geq n > v(x - x^{-1})$. In particular, for those x satisfying (**), we may write the integral in (4.21) as a disjoint union of the form

$$\sum_x |x|^{-1} \eta(x) |x|^{-1} \int_{\varpi^n \mathcal{O}_F} \psi(b(x^{-1} - x)t) dt = 0.$$

Therefore, we have proved that the only possible non-zero contribution comes from those x violating (**). In particular, $|\Psi_C(a)|$ is bounded by the volume of x such that $v(x - x^{-1}) \geq [(d - 1 - v(b))/2]$. It is easy to see that the volume is bounded by $\alpha|b|^{-1/2} = \alpha|a|^{-1/4}$ for some constant α independent of C . □

Remark 14. Just like [57, Proposition VIII.1], the same argument in the proof above actually yields a formula of $\Psi(a)$ and $\Phi(a)$ for large a .

A Davenport–Hasse type relation. We first establish one more lemma.

Lemma 4.11. *When $\frac{3}{4} < \operatorname{Re}(\chi) < 1$, we have*

$$\gamma(\chi, \psi)\gamma(\chi\eta, \psi) = \int_F \Psi(a)\chi^{-1}(a)da,$$

where the integral converges absolutely. Similarly, when $\frac{3}{4} < \operatorname{Re}(\chi) < 1$ we have

$$\gamma(\chi_E, \psi_E) = \int_F \Phi(a)\chi^{-1}(a)da.$$

Proof. By Lemma 4.8, we have for C large enough

$$\gamma(\chi, \psi)\gamma(\chi\eta, \psi) = \int_{|x|<C} \psi(x)\widehat{\chi}(x)d^\times x \int_{|x|<C} \psi(y)\widehat{\chi}\eta(y)d^\times y.$$

This is equal to

$$\begin{aligned} & \int_{|x|,|y|<C} \psi(x+y)\widehat{\chi}(xy)\eta(y)d^\times x d^\times y \\ &= \int_{F^\times} \left(\int_{|a|/C < |y| < C} \psi(a/y+y)\eta(y)d^\times y \right) \widehat{\chi}(a)d^\times a \\ &= \int_{F^\times} \Psi_C(a)d^\times a. \end{aligned}$$

The function $|\Phi_C(a)|$ is bounded by $\beta_1|\log|a|| + \beta_2$ when a is small by Lemma 4.9 and bounded by $\alpha|a|^{1/4}$ when a is large by Lemma 4.10. The first result now follows from Lebesgue’s dominance convergence theorem. Similarly, the second result follows from the fact that $|\Phi(a)|$ is bounded by a constant when $|a|$ is small, and by $\alpha|a|^{1/4}$ when $|a|$ is large. \square

The Davenport–Hasse type relation alluded in the title is as follows.

Theorem 4.12. *We have for all $a \in F^\times$:*

$$\Psi(a) = \Phi(a)\lambda_{E/F}(\psi).$$

Proof. Note that $L(\operatorname{Ind}_{K/F}1_K, s) = L(1_K, s)$. Thus we have an equivalent form of the Langlands constant by (4.12):

$$\lambda_{K/F}(\psi) = \frac{\gamma(\operatorname{Ind}_{K/F}1_K, s, \psi)}{\gamma(1_K, s, \psi_K)}.$$

Now let $K = E$. Then for all characters χ of F^\times such that $\frac{3}{4} < \operatorname{Re}(\chi) < 1$, we have by Lemma 4.11:

$$\int_F \Psi(a)\chi^{-1}(a)da = \lambda_{E/F}(\psi) \int_F \Phi(a)\chi^{-1}(a)da.$$

Both integrals converge absolutely. The theorem now follows easily. \square

4.3 A property under base change

For later use we need a property of the Langlands constant (4.12) under base change. If $K = K_1 \times K_2 \times \dots \times K_m$, we define $\lambda_{K/F}(\psi)$ as the product $\prod_{i=1}^m \lambda_{K_i/F}(\psi)$. Similarly, $\epsilon(\text{Ind}_{K/F} 1_K, 1/2, \psi)$ is by definition $\prod_{i=1}^m \epsilon(\text{Ind}_{K_i/F} 1_{K_i}, 1/2, \psi)$. Recall that for an arbitrary field extension K/F of degree d , we may define a *discriminant* $\delta_{K/F} \in F^\times / (F^\times)^2$ as follows. Choose an F -basis $\alpha_1, \dots, \alpha_d$ of K and let $\sigma_1, \dots, \sigma_d$ be all F -embeddings of K into an algebraic closure of F . Then it is easy to see that

$$(4.22) \quad \det(\sigma_j(\alpha_i))_{1 \leq i, j \leq d}^2 \in F^\times.$$

And if we change the F -basis, this only changes by a square in F^\times . So we define $\delta_{K/F}$ to be the class in $F^\times / (F^\times)^2$ of the determinant (4.22). If $K = F(\alpha)$ and α has minimal polynomial f , then $\delta_{K/F}$ is the class of the discriminant of the polynomial f . In particular, in this case we can choose a representative of $\delta_{K/F}$ such that it lies in $(-1)^{d(d-1)/2} \mathbb{N}_{K/F} K^\times$. Finally, for a quadratic character η of F^\times , it makes sense to evaluate $\eta(\delta_{K/F})$. We extend the definition in the evident way to a product of fields $K = K_1 \times K_2 \times \dots \times K_m$.

The property we need is the following.

Theorem 4.13. *Let E/F be a quadratic field extension and let F'/F be a field extension of degree d . Let $E' = E \otimes_F F'$. Then we have*

$$(4.23) \quad \lambda_{E'/F'}(\psi_{F'}) = \lambda_{E/F}^d(\psi_F) \eta_{E/F}(\delta_{F'/F}).$$

Proof. First of all we have the following simple observation. If we replace ψ by ψ_a , $a \in F^\times$, the right hand side of (4.23) changes by a factor $\eta_{E/F}(a)^d$ and the left hand side of (4.23) changes by a factor $\eta_{E'/F'}(a) = \eta_{E/F}(\mathbb{N}_{F'/F} a) = \eta_{E/F}(a^d)$. Therefore it suffices to prove the identity for any one choice of ψ .

Note that (4.23) is equivalent to

$$\epsilon(\eta_{E'/F'}, 1/2, \psi_{F'}) = \epsilon(\eta_{E/F}, 1/2, \psi_F)^d \cdot \eta_{E/F}(\delta_{F'/F}).$$

Lemma 4.14. *If E/F is unramified, then (4.23) holds.*

Proof. We may choose ψ to have conductor equal to zero. Since $\eta_{E/F}$ is unramified, we have

$$\epsilon(\eta_{E/F}, 1/2, \psi_F) = 1.$$

Now $\psi_{F'}$ has conductor denoted by k which is equal to the valuation of the different $\mathcal{D}_{F'/F}$, namely $\mathcal{D}_{F'/F} = (\varpi_{F'})^k$. Let e be the ramification index and $f = d/e$.

Case I: f is even. Then $E \subset F'$ is a subfield. In particular, $\mathbb{N}_{F'/F} F'^{\times} \subset \mathbb{N}_{E/F} E^\times$. In this case, the left hand side of (4.23) is equal to 1 as $\eta_{E'/F'}$ is trivial. As we may choose a representative of $\delta_{F'/F}$ in $(-1)^{d(d-1)/2} \mathbb{N}_{K/F} K^\times \subset \mathbb{N}_{E/F} E^\times$, we see that $\eta_{E/F}(\delta_{F'/F}) = 1$.

Case II: f is odd. Then $\epsilon(\eta_{E/F}, 1/2, \psi_F) = (-1)^k$. Note that $\mathbb{N}_{F'/F}(\varpi_{F'}) \in (\varpi_F)^f \mathcal{O}_F^\times$. So we have $\mathbb{N}_{F'/F} \mathcal{D}_{F'/F} = (\varpi_F)^{kf}$. Since f is odd, kf and k have the same parity. Since the valuation of $\mathbb{N}_{F'/F} \mathcal{D}_{F'/F}$ has the same parity as the valuation of $\delta_{F'/F}$, we see that $\eta_{E/F}(\delta_{F'/F}) = (-1)^{kf} = (-1)^k$ as desired. \square

Lemma 4.15. *If E/F is archimedean, then (4.23) holds.*

Proof. The case $F = \mathbb{C}$ is obvious. Now we assume that $F = \mathbb{R}$ and $E = \mathbb{C}$. Then the only non-trivial case we need to consider is when $F' = \mathbb{C}$. Then the left hand side is equal to one. For the right hand side, we have

$$\lambda_{E/F}(\psi)^2 = \eta_{E/F}(-1) = -1.$$

And

$$\eta_{E/F}(D_{F'/F}) = -1.$$

This completes the proof. \square

We now treat the general case for E/F . We use a global argument. To do so we want to globalize the quadratic extension E/F . We use the following lemma from [15, §14].

Lemma 4.16. *Let E/F be a quadratic extension of non-archimedean local fields. Then there exists a totally real number field \mathcal{F} with F as its completion at a place v_0 of \mathcal{F} , and a quadratic totally imaginary extension \mathcal{E} of \mathcal{F} with corresponding completion E at v_0 such that \mathcal{E} is unramified over \mathcal{F} at all finite places different from v_0 .*

Since the global epsilon factor satisfies

$$\epsilon(\text{Ind}_{\mathcal{E}/\mathcal{F}} 1_{\mathcal{E}}, s, \psi_{\mathcal{F}}) = \epsilon(1_{\mathcal{E}}, s, \psi_{\mathcal{E}}),$$

we have a product formula

$$\prod_v \lambda_{\mathcal{E}_v/\mathcal{F}_v}(\psi_v) = 1,$$

where \mathcal{E}_v is a product of field extensions of \mathcal{F}_v and v runs over all places of \mathcal{F} . We choose a finite extension \mathcal{F}' of \mathcal{F} such that v_0 is inert and $\mathcal{F}'_{v_0} \simeq F'$. Such choice obviously exists. Then it is clear that (4.23) holds for all $\mathcal{E}_v/\mathcal{F}_v$ at those $v \neq v_0$. By the global identity and $\prod_v \eta_{\mathcal{E}_v/\mathcal{F}_v}(\delta_{\mathcal{F}'/\mathcal{F}}) = 1$, we immediately deduce (4.23) at the place v_0 . This completes the proof of Theorem 4.13. \square

4.4 All Fourier transforms preserve transfer

Now we need to consider simultaneously the general linear case and the unitary case. Let E/F be a fixed quadratic field extension. We set up some notations. We will use \mathcal{V}' to denote $\mathfrak{gl}_n(F) \times F^n \times F_n$. There are two isomorphism classes of Hermitian spaces of dimension n , which we will denote by W_1, W_2 respectively. Then we let \mathcal{V}_i denote $\mathfrak{U}(W_i) \times W_i$ for $i = 1, 2$.

Note that we have an obvious way to match the partial Fourier transforms on \mathcal{V}' and \mathcal{V}_i . Recall that all Haar measures to define Fourier transform are chosen to be self-dual.

Theorem 4.17. *For any a fixed Fourier transform \mathcal{F} , there exists a constant $\nu \in \mu_4$ depending only on $n, \psi, E/F$ and the Fourier transform \mathcal{F} with the following property: if $f \in \mathcal{C}_c^\infty(\mathcal{V})$ and the pair $(f_i \in \mathcal{C}_c^\infty(\mathcal{V}_i))_{i=1,2}$ are smooth transfer of each other, so are $\nu \mathcal{F}(f)$ and $(\mathcal{F}(f_i))_{i=1,2}$.*

Proof. We now let \mathcal{F}_a ($\mathcal{F}_b, \mathcal{F}_c$, resp.) be the Fourier transform with respect to the total space (the subspace $\mathfrak{U}(W)$, W , resp.). Consider the following three statements:

- A_n : For all E/F , ψ , and W_i of dimension n , there is a constant $\nu \in \mu_4$ with the property: if $f \in \mathcal{C}_c^\infty(\mathcal{V})$ and $f_i \in \mathcal{C}_c^\infty(\mathcal{V}_i)$ ($i = 1, 2$) match, then so do $\nu \mathcal{F}_a(f)$ and $\mathcal{F}_a(f_i)$.
- B_n : For all E/F , ψ , and W_i of dimension n , there is a constant $\nu \in \mu_4$ with the property: if $f \in \mathcal{C}_c^\infty(\mathcal{V})$ and $f_i \in \mathcal{C}_c^\infty(\mathcal{V}_i)$ ($i = 1, 2$) match, then so do $\nu \mathcal{F}_b(f)$ and $\mathcal{F}_b(f_i)$.
- C_n : For all E/F , ψ , and W_i of dimension n , if $f \in \mathcal{C}_c^\infty(\mathcal{V})$ and $f_i \in \mathcal{C}_c^\infty(\mathcal{V}_i)$ ($i = 1, 2$) match, then so do $\lambda_{E/F}(\psi)^{-n} \mathcal{F}_c(f)$ and $\mathcal{F}_c(f_i)$.

The theorem follows immediately if we prove the following three claims:

1. $A_{n-1} \Rightarrow B_n$.
2. $C_1 \Leftrightarrow C_n$.
3. $B_n + C_n \Rightarrow A_n$.

The proofs the three claims are provided below in Lemma 4.18, 4.19, 4.21, respectively. \square

Lemma 4.18. $A_{n-1} \Rightarrow B_n$.

Proof. We now use \hat{f} to denote the Fourier transforms with respect to \mathfrak{gl}_n and $\mathfrak{U}(W)$. We first consider the general linear side. We let $W = F_n \times F^n$ and consider it as an $F \times F$ -module of rank n with the Hermitian form $(w, w) = uv$ if $w = (u, v) \in F_n \times F^n$. We also denote the normalized orbital integral

$$(4.24) \quad O_{X,w}^\eta(f) = \omega(X, w) O_{X,w}(f), \quad (X, w) \in \mathcal{V}',$$

where $\omega(X, w)$ is the transfer factor (3.7), and $O_{X,w}(f)$ is defined by (4.5).

By the local trace formula Theorem 4.6, for $w \in W$ with $(w, w) \neq 0$, $f \in \mathcal{C}_c^\infty(\mathcal{V})$ and $g \in \mathcal{C}_c^\infty(\mathfrak{gl}_n(F))$, we have:

$$(4.25) \quad \int_{\mathfrak{gl}_n(F)} O_{X,w}^\eta(f) \omega(X, w) g(X) dX = \int_{\mathfrak{gl}_n(F)} O_{X,w}^\eta(\hat{f}) \omega(X, w) \check{g}(X) dX.$$

Let $(w)^\perp$ be the orthogonal complement of $(F \times F)w$ in W . Up to the $H = \mathrm{GL}_{n,F}$ -action, we may choose w of the form (e, de') where $e = (0, \dots, 0, 1)$ and $d \in F^\times$ is the Hermitian norm (w, w) . Then the stabilizer of w in H can be identified with $\mathrm{GL}_{n-1,F}$ (with the embedding into $\mathrm{GL}_{n,F}$ as before). If $h \in \mathrm{GL}_{n-1}(F)$, we have $O_{X,w}^\eta(f) = O_{X^h,w^h}^\eta(f) = O_{X^h,w}^\eta(f)$. Then we may rewrite the left hand side of (4.25) as

$$\int_{\mathfrak{gl}_n(F)} O_{X,w}^\eta(f) \omega(X, w) g(X) dX = \int_{Q(F)} O_{X,w}^\eta(f) \left(\int_{\mathrm{GL}_{n-1}(F)} \omega(X^h, w) g(X^h) dh \right) dq(X),$$

where $Q_{n-1} = \mathfrak{gl}_n // \mathrm{GL}_{n-1}$, q is the quotient morphism and the measure is a suitable one on $Q(F)$ such that for all $g \in \mathcal{C}_c^\infty(\mathfrak{gl}_n(F))$:

$$\int_{\mathfrak{gl}_n(F)} g(X) dX = \int_{Q(F)} \left(\int_{\mathrm{GL}_{n-1}(F)} g(X^h) dh \right) dq(X).$$

Note that $\omega(X^h, w) = \eta(h)\omega(X, w)$ for $h \in \mathrm{GL}_{n-1}(F)$. Now it is easy to see that when we restrict the transfer factor $\omega(X, w)$ to $X \in \mathfrak{gl}_n$, it is a constant multiple of the transfer factor we have used to define the $\mathrm{GL}_{n-1, F}$ -orbital integral. Moreover, this constant depends only on w and is denoted by c_w . So we have

$$\int_{\mathrm{GL}_{n-1}(F)} \omega(X^h, w)g(X^h)dh = \omega(X, w) \int_{\mathrm{GL}_{n-1}(F)} g(X^h)\eta(h)dh = c_w O_X^\eta(g).$$

Then this depends only on the $\mathrm{GL}_{n-1}(F)$ -orbit of X .

We have a similar result for the right hand side of (4.25). The constant c_w is then canceled. Replacing g by \hat{g} , we deduce from (4.25) that

$$(4.26) \quad \int_{Q(F)} O_{X,w}^\eta(f)O_X^\eta(\hat{g})dq(X) = \int_{Q(F)} O_{X,w}^\eta(\hat{f})O_X^\eta(g)dq(X).$$

Note that the Fourier transform here is \mathcal{F}_b for GL_n -action on \mathcal{V}' but it is the \mathcal{F}_a for the GL_{n-1} -action on \mathfrak{gl}_n .

In the unitary case we also have a similar equality for $i = 1, 2$ (without the issue of transfer factors)

$$\int_{Q(F)} O_{X,w}(f_i)O_X(\hat{g}_i)dq(X) = \int_{Q(F)} O_{X,w}(\hat{f}_i)O_X(g_i)dq(X).$$

Here the stabilizer of w in $\mathfrak{U}(W)$ replaces $\mathrm{GL}_{n-1, F}$. Note that we have identified the categorical quotient Q with Q_i as before.

Now suppose that $f \leftrightarrow (f_i)$. We want to show that for some constant ν :

$$(4.27) \quad \nu O_{X^0, w^0}^\eta(\hat{f}) = O_{X_i^0, w_i^0}(\hat{f}_i)$$

for any *strongly* regular semisimple $(X^0, w^0) \leftrightarrow (X_i^0, w_i^0)$. This would imply the equality for all matching regular semisimple elements by the local constancy of orbital integrals (Lemma 3.12).

We may choose $g \leftrightarrow (g_i)$ such that

- Both are supported in the regular semisimple locus.
- There exists a small (open and compact) neighborhood \mathcal{U} of $q(X^0) \in Q(F)$ with the following property: (1) the functions on \mathcal{U} given by $q(X) \mapsto O_{X, w^0}^\eta(\hat{f})$ and $q(X_i) \mapsto O_{X_i, w_i^0}(\hat{f}_i)$ are constant; (2) the functions on $Q(F)$ given by $q(X) \mapsto O_X^\eta(g)$ and $q(X_i) \mapsto O_X(g_i)$ are the characteristic function $1_{\mathcal{U}}$.

This is clearly possible by Lemma 3.12. For such a choice we have

$$\int_Q O_{X, w^0}^\eta(f)O_X^{\eta_i}(\hat{g})dq(X) = O_{X^0, w_i^0}^\eta(\hat{f}) \left(\int_{\mathcal{U}} dq(X) \right), \quad i = 1, 2.$$

Now by A_{n-1} , we have for some constant ν independent of g, g_i

$$\nu O_X^\eta(\hat{g}) = O_{X_i}(\hat{g}_i)$$

whenever $X \leftrightarrow X_i$. Now the desired equality (4.27) follows immediately for the same constant ν as in A_{n-1} . This proves that $A_{n-1} \Rightarrow B_n$. \square

Lemma 4.19. $C_1 \Leftrightarrow C_n$.

Proof. It suffices to show $C_1 \Rightarrow C_n$. Now we will use \widehat{f} to denote the Fourier transform with respect to $W = F_n \times F^n$ and W_i respectively. We want to show that if f and f_i match, then for strongly regular $(X, w) \leftrightarrow (X_i, w_i)$, we have

$$(4.28) \quad \lambda_{E/F}(\psi)^n O_{X,w}^\eta(\widehat{f}) = O_{X_i,w_i}(\widehat{f_i}).$$

For a regular semisimple $X \in \mathfrak{gl}_n(F)$, let T be the centralizer of X in H and \mathfrak{t} its Lie algebra. Then T is isomorphic to $\prod_j F_j^\times$ for some field extensions F_j/F of degree n_j with $\sum_{j=1}^r n_j = n$. For a regular semisimple $X_i \in \mathfrak{U}(W_i)$, let T_i be the centralizer of X_i in H_i and \mathfrak{t}_i its Lie algebra. Let $E_j = E \otimes_F F_j$. As X, X_i have the same characteristic polynomial, we know that T_i is isomorphic to $\prod_j \text{Res}_{F_j/F} E_j^1$ for the same tuple of field extensions F_j/F . Here E^1 is the kernel (as an algebraic group) of norm homomorphism $N_{E/F} : E^\times \rightarrow F^\times$. Let $F' := \prod_i F_j$, $E' := F' \otimes_F E$. By [3, §5], W is a rank one Hermitian space over E' with unitary group $U(W, E'/F') \simeq T$. We may identify F' with the sub-algebra $F[X] \subset \mathfrak{gl}_n$.

For a more intrinsic exposition, we let M (M^* , resp.) denote F_n (F^n , resp.) and $\mathfrak{gl}_n = \text{End}(M)$. We may describe the transfer factor as follows:

$$\omega(X, u, v) = \eta \left(\frac{u \wedge Xu \wedge X^2 u \dots \wedge X^{n-1} u}{\omega_0} \right), X \in \text{End}(M), u \in M, v \in M^*,$$

where ω_0 a fixed generator of the F -line $\wedge_F^n M$. If we change the generator ω_0 , the transfer factor only changes by a constant in $\{\pm 1\}$.

Then under the action of $F' = F[X]$, M is a free F' -module of rank one. In this way, $M^* = \text{Hom}_F(M, F)$ is canonically isomorphic to $\text{Hom}_{F'}(M, F')$. Indeed, we may define an F' -linear pairing $(\cdot, \cdot)_{F'} : M \times M^* \rightarrow F'$ such that for all $\lambda \in F', x \in M, y \in M^*$ we have

$$\text{tr}_{F'/F}(\lambda x, y)_{F'} = (\lambda x, y)_F.$$

Fixing a generator of $\wedge_F^n M \simeq F'$ we define a transfer factor $\omega(w) \in \{\pm 1\}$ corresponding to the case $n = 1$. We also have a compatibility $\eta_{E/F}(N_{F'/F} x) = \eta_{E'/F'}(x)$ and $N_{F'/F} x = \det(x)$ when $x \in F' = F[X]$.

We then have an inversion formula as follows.

Lemma 4.20. *For a regular semisimple $X \in \mathfrak{gl}_n(F)$ with centralizer $T \simeq \prod_j \text{Res}_{F_j/F} \text{GL}_1$, let $\widehat{\kappa}^\eta(w, w')$ be the locally constant $T \times T$ -invariant function on $W_{rs} \times W_{rs} \rightarrow \mathbb{C}$ given by Corollary 4.7. Then we have*

$$\widehat{O}_{X,w}^\eta(f) = \eta_{E/F}(\delta_{F'/F}) \int_{Q(F')} O_{X,w'}^\eta(f) \kappa^\eta(w, w') dq(w'),$$

where $Q = (M \times M^*) // \text{Res}_{F'/F} \text{GL}_1$.

Proof. Without loss of generality, we may assume that $f = \phi \otimes \varphi$, $\phi \in \mathcal{C}_c^\infty(\mathfrak{gl}_n(F))$, $\varphi \in \mathcal{C}_c^\infty(M \times M^*)$. We now see that the orbital integral can be rewritten as

$$\begin{aligned} O_{X,w}^\eta(\phi \otimes \widehat{\varphi}) &= \omega(X, w) / \omega(w) \int_{T \setminus H} \phi(X^h) \eta(h) O_w^\eta(h \widehat{\varphi}) dh \\ &= \omega(X, w) / \omega(w) \int_{T \setminus H} \phi(X^h) \eta(h) O_w^\eta(\widehat{h \varphi}) dh, \end{aligned}$$

where we write ${}^h\varphi(w) = \varphi(w^{h^{-1}})$.

By Corollary 4.7, we have

$$(4.29) \quad O_w^\eta(\hat{\varphi}) = \int_{Q(F')} O_{w'}^\eta(\varphi) \kappa^\eta(w, w') dq(w').$$

Reversing the argument we obtain

$$O_{X,w}^\eta(\phi \otimes \hat{\varphi}) = \omega(X, w) \omega(w) \int_{Q(F')} \omega(X, w') \omega(w') O_{X,w'}^\eta(\phi \otimes \varphi) \kappa^\eta(w, w') dq(w').$$

It is easy to verify that

$$\omega(X, w) \omega(X, w') = \eta(\delta_X) \omega(w) \omega(w'),$$

where $\delta(X)$ is the discriminant of the characteristic polynomial of X . In particular, $\eta(\delta_X) = \eta(\delta_{F'/F})$. Note that the product $\omega(X, w) \omega(X, w')$ ($\omega(w) \omega(w')$, resp.) does not depend on the choice of the generator of the F -line $\wedge_F^n M$ (the F' -line $\wedge_{F'}^1 M = M$, resp.). This completes the proof. \square

Similarly we also have an inversion formula in the unitary case with a different kernel function denoted by $\kappa_i(w_i, w'_i)$. Finally we note that the kernel functions are given by the Kloosterman sums relative to E'/F' . By Theorem 4.12, we have

$$(4.30) \quad \kappa^\eta(w, w') = \lambda_{E'/F'}(\psi_{F'}) \kappa_i(w_i, w'_i)$$

whenever $w \leftrightarrow w_i$ and $w' \leftrightarrow w'_i$, $i = 1, 2$.

Now the proof of $C_1 \Rightarrow C_n$ follows from the inversion formulae, the relation between the kernel functions (4.30) and the base change property of the Langlands constant (Theorem 4.13):

$$\lambda_{E'/F'}(\psi_{F'}) = \lambda_{E/F}^n(\psi_F) \eta(\delta_{F'/F}).$$

\square

Remark 15. It is easy to see that for fixed X, X_i , the statement C_1 implies that, up to a constant multiple, the partial Fourier transform have matching orbital integrals for those elements with first components X, X_i . The lengthy computation of the Davenport–Hasse relations is to show that this constant, a priori depending on X, X_i , is indeed independent of the choice of X, X_i .

Lemma 4.21. $B_n + C_n \Rightarrow A_n$.

Proof. This is obvious since $\mathcal{F}_a = \mathcal{F}_b \mathcal{F}_c$. \square

4.5 Completion of the proof of Theorem 2.6

In §3, we have reduced the existence of transfer on groups to the Lie algebra version: by Theorem 3.3, it suffices to show Conjecture \mathfrak{S}_{n+1} . Obviously, Conjecture \mathfrak{S}_{n+1} is equivalent to the corresponding assertion on the following subspaces:

$$\mathfrak{sl}_n \times F_n \times F^n$$

and for Hermitian W_i , $i = 1, 2$:

$$\mathfrak{sl}(W_i) \times W_i.$$

Here \mathfrak{sl}_n ($\mathfrak{sl}(W_i)$, resp.) denotes the subspace of \mathfrak{gl}_n ($\mathfrak{U}(W_i)$, resp.) of trace-zero elements.

We let \mathcal{V} be either $\mathfrak{sl}_n(F) \times F^n \times F_n$ or $\mathfrak{sl}(W_i) \times W_i$ and let η be the quadratic character associated to E/F in the former case and the trivial character in the latter. For a distribution T on \mathcal{V} , we denote by $\hat{T}^\mathcal{V}$, \hat{T}^W , $\hat{T}^{\mathfrak{U}(W)}$ the three partial Fourier transforms respectively. Similar notation applies to functions f on \mathcal{V} .

The following homogeneity result enables us to deduce the existence of transfer from the compatibility with Fourier transform.

Theorem 4.22 (Aizenbud). *There is no distribution T on \mathcal{V} satisfying both of the following properties*

- (1) T is (H, η) -invariant (hence so is \hat{T}).
- (2) T , $\hat{T}^\mathcal{V}$, \hat{T}^W , $\hat{T}^{\mathfrak{U}(W)}$ are all supported in the nilpotent cone \mathcal{N} .

Proof. This is proved in [1, Theorem 6.2.1] for the case $\eta = 1$. But the same proof goes through for the nontrivial quadratic η . □

Corollary 4.23. *Set*

$$\mathcal{C}_0 = \cap_T \text{Ker}(T) \subset \mathcal{C}_c^\infty(\mathcal{V}),$$

where T runs over all (H, η) -invariant distributions on \mathcal{V} . Then the space $\mathcal{C}_c^\infty(\mathcal{V})$ is the sum of \mathcal{C}_0 and the image of all Fourier transforms of $\mathcal{C}_c^\infty(\mathcal{V} - \mathcal{N})$. Equivalently, any $f \in \mathcal{C}_c^\infty(\mathcal{V})$ can be written as

$$f = f_0 + f_1 + \hat{f}_2^\mathcal{V} + \hat{f}_3^W + \hat{f}_4^{\mathfrak{U}(W)},$$

where $f_0 \in \mathcal{C}_0$, $f_i \in \mathcal{C}_c^\infty(\mathcal{V} - \mathcal{N})$, $i = 1, 2, 3$.

Proof. Let \mathcal{C} be the subspace spanned by \mathcal{C}_0 and the image of all Fourier transforms of $\mathcal{C}_c^\infty(\mathcal{V} - \mathcal{N})$. If the quotient $L = \mathcal{C}_c^\infty(\mathcal{V})/\mathcal{C}$ is not trivial, then there must exist a nontrivial linear functional on L . This induces a nonzero distribution T on \mathcal{V} . As T is zero on \mathcal{C}_0 , T is (H, η) -invariant. As T is zero on $\mathcal{C}_c^\infty(\mathcal{V} - \mathcal{N})$, it is supported on \mathcal{N} . Similarly, all Fourier transforms $\hat{T}^\mathcal{V}$, \hat{T}^W , $\hat{T}^{\mathfrak{U}(W)}$ are (H, η) -invariant and supported on \mathcal{N} . This contradicts Theorem 4.22. □

Finally we return to prove Theorem 2.6.

Proof. By abuse of notation, we still denote by $Q = \mathbb{A}^{2n}$ the categorical quotient of \mathcal{V} by H . By the localization principle Proposition 3.8, it is enough to prove the existence of local transfers at all $z \in Q(F)$ (not necessarily regular semisimple elements). We will prove this by induction on n . If $z \in Q(F)$ is non-zero, the stabilizer of z is strictly smaller than H . By Proposition 3.16, the local transfer around z is implied by the local transfer around 0 for the sliced representations. The sliced representations are (possibly a product) of the same type with smaller dimension (with possibly a base change of the base field F). Then by induction hypothesis, we may assume the existence of local transfers at all non-zero semisimple $z \in Q(F)$. Therefore, by the localization principle Proposition 3.8 (or, what its proof shows), we know the existence of smooth transfer for functions supported away from the nilpotent cone. By Theorem 4.17 on the compatibility with Fourier transforms, this implies the existence of smooth transfer for functions \hat{f} where f is supported away from the nilpotent cone and \hat{f} is one of the three Fourier transforms. By Corollary 4.23, we have proved the existence of smooth transfer for all functions f (those in \mathcal{V}_0 clearly admit smooth transfers). This completes the proof of Theorem 2.6. \square

A Spherical characters for a strongly tempered pair.

By Atsushi Ichino and Wei Zhang

2

Let F be a non-archimedean local field. We consider a pair (G, H) where G is a reductive group and H is a subgroup. We will also denote by G, H the sets of F -points. We will assume that H a spherical subgroup in the sense that $X := G/H$ with the G -action from left is a spherical variety. Following [54, sec. 6], we say that the pair (G, H) is *strongly tempered* if for any tempered unitary representation π of G , and any $u, v \in \pi$, the associated matrix coefficient $\phi_{u,v}(g) := \langle \pi(g)u, v \rangle$ ($\langle \cdot, \cdot \rangle$ is the Hermitian G -invariant inner product) satisfies

$$\phi_{u,v}|_H \in L^1(H).$$

To check whether a pair (G, H) is strongly tempered, one uses the Harish-Chandra function Ξ . Let π_0 be the normalized induction of the trivial representation of a Borel B of G and let v_0 be the unique spherical vector such that $\langle v_0, v_0 \rangle = 1$. Then Ξ is the matrix coefficient:

$$\Xi(g) = \langle gv_0, v_0 \rangle.$$

We have $\Xi(g) \geq 0$ for any $g \in G$. Then (G, H) is strongly tempered if $\Xi|_H \in L^1(H)$.

The major examples are those appearing in the Gan–Gross–Prasad conjecture:

- Let V be an orthogonal space of dimension $n + 1$ and W a codimension one subspace (both non-degenerate). Let $\mathrm{SO}(V)$ and $\mathrm{SO}(W)$ be the corresponding special orthogonal groups. Let $G = \mathrm{SO}(W) \times \mathrm{SO}(V)$, and let $H \subset G$ be the graph of the embedding $\mathrm{SO}(W) \hookrightarrow \mathrm{SO}(V)$ induced by $W \hookrightarrow V$.
- $G = \mathrm{GL}_n \times \mathrm{GL}_{n+1}$, H is the graph of the embedding of $\mathrm{GL}_n \hookrightarrow \mathrm{GL}_{n+1}$ given by $g \mapsto \mathrm{diag}(g, 1)$.
- Let E/F be a quadratic extension of fields. Let V be a Hermitian space of dimension $n + 1$ and W a codimension one subspace (both non-degenerate). Let $U(V)$ and $U(W)$ the corresponding unitary groups. Let $G = U(W) \times U(V)$, and let $H \subset G$ be the graph of the embedding $U(W) \hookrightarrow U(V)$ induced by $W \hookrightarrow V$.

A proof of the fact that these pair (G, H) are strongly tempered can be found in [31] for the orthogonal case and [26] for the linear and unitary cases.

Assume that (G, H) are strongly tempered. Then for any tempered representation π of G , following [31] we may define a matrix coefficient integration

$$\nu(u, v) := \int_H \langle \pi(h)u, v \rangle dh.$$

Obviously $\nu \in \mathrm{Hom}_{H \times H}(\pi \otimes \tilde{\pi}, \mathbb{C})$. The integral is absolutely convergent by the strong temperedness.

The following is a conjecture of Ichino–Ikeda in their refinement of the Gan–Gross–Prasad conjecture.

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Theorem A.1 (Sakellaridis–Venkatesh, [54]). *Assume that (G, H) is strongly tempered and $X = G/H$ is wavefront. Let π be an irreducible representation. Assume that π is a discrete series representation or $\pi = \text{Ind}_P^G(\sigma)$ for a discrete series representation σ of the Levi of a parabolic subgroup P . Then $\text{Hom}_H(\pi, \mathbb{C}) \neq 0$ if and only if ν does not vanish identically.*

The result is also proved in the orthogonal case by Waldspurger.

For the rest of the appendix, let π be a *tempered* representation. We further assume that

$$\dim \text{Hom}_H(\pi, \mathbb{C}) \leq 1.$$

All the examples we list earlier satisfies this condition.

Let $\ell \in \text{Hom}_H(\pi, \mathbb{C})$. Then we define a *spherical character* associated to ℓ to be a distribution on G such that

$$\theta_{\pi, \ell}(f) := \sum_{v \in \mathcal{B}(\pi)} \ell(\pi(f)v) \overline{\ell(v)}, \quad f \in \mathcal{C}_c^\infty(G),$$

where $\mathcal{B}(\pi)$ is an orthonormal basis of π . The distribution is bi- H -invariant for the left and right translation by H . Obviously the distribution $\theta_{\pi, \ell}$ is non-zero if and only if the linear functional ℓ is non-zero.

This note is to prove the following:

Theorem A.2. *Assume that $\ell \neq 0$. Fix any open dense subset G_r of G . Then the restriction of the distribution $\theta_{\pi, \ell}$ to G_r is non-zero. Equivalently, there exists $f \in \mathcal{C}_c^\infty(G_r)$ such that $\theta_{\pi, \ell}(f) \neq 0$.*

We make some preparation first. By Theorem of Sakellaridis–Venkatesh above, there exist $v_0 \in \pi$ such that

$$\ell(v) = \int_H \langle \pi(h)v, v_0 \rangle dh, \quad v \in \pi.$$

Lemma A.3. *For all $f \in \mathcal{C}_c^\infty(G)$, we have*

$$\theta_{\pi, \ell}(f) = \int_H \int_H \left(\int_G f(g) \Phi(h_2 g h_1) dg \right) dh_1 dh_2,$$

where

$$(A.1) \quad \Phi(g) = \langle \pi(g)v_0, v_0 \rangle, \quad g \in G.$$

Proof. The proof is analogous to that of [46, Thm. 6.1]. We may rewrite θ as

$$\begin{aligned} \theta_{\pi, \ell}(f) &= \sum_{v \in \mathcal{B}(\pi)} \ell(\pi(f)v) \overline{\ell(v)} \\ &= \sum_{v \in \mathcal{B}(\pi)} \int_H \langle v, \pi(f^*)\pi(h^{-1})v_0 \rangle dh \int_H \langle \pi(h^{-1})v_0, v \rangle dh. \end{aligned}$$

By $\varphi = \sum_{v \in \mathcal{B}(\pi)} \langle \varphi, v \rangle v$ for any $\varphi \in \pi$, we have

$$\begin{aligned} \theta_{\pi, \ell}(f) &= \sum_{v \in \mathcal{B}(\pi)} \int_H \int_H \langle \pi(h_1^{-1})v_0, \pi(f^*)\pi(h_2^{-1})v_0 \rangle dh_1 dh_2 \\ &= \int_H \int_H \left(\int_G f(g) \langle \pi(h_2 g h_1)v_0, v_0 \rangle dg \right) dh_1 dh_2. \end{aligned}$$

□

We consider the orbital integral of the matrix coefficient Φ as in Lemma A.3:

$$(A.2) \quad O(g, \Phi) = \int_H \int_H \Phi(h_1 g h_2) dh_1 dh_2,$$

as well as the orbital integral $O(g, \Xi)$ of the Harish-Chandra function Ξ . The following lemma shows that the integral (A.2) converges for all but a measure-zero set of $g \in G$.

Lemma A.4. *The function $g \mapsto O(g, \Xi)$ on G is locally L^1 . Equivalently, for any $f \in \mathcal{C}_c^\infty(G)$, the following integral is absolutely convergent*

$$\int_G |f(g)O(g, \Xi)| dg < \infty.$$

Proof. Consider a special maximal compact open subgroup K of G such that we have the following relation for a suitable measure on K :

$$(A.3) \quad \int_K \Xi(gkg') dk = \Xi(g)\Xi(g').$$

Such K exists ([53]). Without loss of generality, we may assume that f is the characteristic function of KgK for some $g \in G$. We then have

$$\int_G |f(g)O(g, \Xi)| dg = \int_K \int_K \int_H \int_H \Xi(h_1 k_1 g k_2 h_2) dh_1 dh_2 dk_1 dk_2.$$

By (A.3) this is equal to

$$\left(\int_K \int_H \Xi(h_1 k_1 g) dh_1 dk_1 \right) \left(\int_H \Xi(h_2) dh_2 \right).$$

By (A.3) again, we obtain

$$\Xi(g) \left(\int_H \Xi(h_1) dh_1 \right) \left(\int_H \Xi(h_2) dh_2 \right) < \infty.$$

By Fubini theorem, this shows

$$\int_G |f(g)O(g, \Xi)| dg < \infty.$$

□

Lemma A.5. *Let Φ be the matrix coefficient in (A.1). The function $g \mapsto O(g, \Phi)$ on G is locally L^1 and for any $f \in \mathcal{C}_c^\infty(G)$, we have*

$$\theta_{\pi, \ell}(f) = \int_G f(g) O(g, \Phi) dg.$$

Proof. For any tempered representation π of G and a matrix coefficient $\phi_{u, v}$ associated to $u, v \in \pi$, we have

$$|\phi_{u, v}(g)| \leq c \Xi(g), \quad g \in G$$

for some constant $c > 0$. By Lemma A.4, the function $g \mapsto f(g) O(g, \Xi)$ is in $L^1(G)$. In particular, this implies that the triple integral

$$\int_G \int_H \int_H |f(g)| \Xi(h_2 g h_1) dh_1 dh_2 dg$$

is absolutely convergent. By Fubini theorem we may interchange the order of integration in the following:

$$\int_G \int_H \int_H f(g) \Phi(h_2 g h_1) dh_1 dh_2 dg = \int_H \int_H \left(\int_G f(g) \Phi(h_2 g h_1) dg \right) dh_1 dh_2.$$

This is equal to $\theta_{\pi, \ell}(f)$ by Lemma A.3. □

Now we return to prove Theorem A.2. Since $\theta_{\pi, \ell}$ is non-zero, there exists $f \in \mathcal{C}_c^\infty(G)$ such that

$$\theta_{\pi, \ell}(f) \neq 0.$$

Equivalently, by Lemma A.5,

$$\int_G f(g) O(g, \Phi) dg \neq 0.$$

As G_r is open and dense, we may choose $f_n \in \mathcal{C}_c^\infty(G_r)$, such that point-wisely on G_r we have

$$\lim_{n \rightarrow \infty} f_n = f.$$

Without loss of generality, we may assume that $f \geq 0$ point-wisely and $f - f_n \geq 0$ point-wisely. Then we have

$$|(f(g) - f_n(g)) O(g, \Phi)| \leq 2f(g) |O(g, \Phi)|$$

which is integrable on G by Lemma A.4. By Lebesgue's dominated convergence theorem, we have

$$\lim_{n \rightarrow \infty} \int_G (f(g) - f_n(g)) O(g, \Phi) dg = 0.$$

Since $\int_G f(g) O(g, \Phi) dg \neq 0$, we have for n large enough,

$$\int_G f_n(g) O(g, \Phi) dg \neq 0,$$

or equivalently,

$$\theta_{\pi, \ell}(f_n) \neq 0.$$

As $f_n \in \mathcal{C}_c^\infty(G_r)$, this completes the proof of Theorem A.2.

Remark 16. When comparing with the property of the usual character as established by Harish-Chandra, some questions remain for the spherical characters in this notes. For example, is the function $g \mapsto O(g, \Phi)$ continuous or even locally constant on an open dense subset? If π is super-cuspidal, one may prove that the function $g \mapsto O(g, \Phi)$ is locally constant on some open dense subset in the unitary case listed in the examples.

B Explicit étale Luna slices

In this appendix, we construct étale Luna slices explicitly (Theorem B.5). We will first describe the sliced representations at a semisimple element of \mathfrak{S} or $\mathfrak{U}(V)$. Then we exhibit an étale Luna slice at x . Some of the key construction is already in [51]. But we need to show that their construction actually gives étale Luna slices. The steps are close to the Harish-Chandra's descent method (cf. [40]).

Sliced representations at semisimple elements. We start with $\mathfrak{U}(V)$ where $V = W \oplus Eu$. It suffices to consider $\mathfrak{U}(W) \times W$. We may write an element in $\mathfrak{U}(W) \times W$ as (X, w) , $X \in \mathfrak{U}(W)$, $w \in W$. Denote by W_2 the subspace of W generated by $X^i w$, $i \geq 0$. By [51, Theorem 17.2], for (X, w) to be semisimple, it is necessary that W_2 is a non-degenerate subspace. In this case, we have an orthogonal decomposition $W = W_1 \oplus W_2$. Then X stabilizes both subspaces and we may write $X = \text{diag}[X_1, X_2]$ for $X_i \in \mathfrak{U}(W_i)$. Then (X, w) is semisimple if and only if X_2 is semisimple (in the usual sense) in $\mathfrak{U}(W_2)$. It is also self-evident that (X_2, w) defines a regular semisimple element relative to the action of $U(W_2)$ on $\mathfrak{U}(W_2) \times W_2$. Then the stabilizer of (X, w) is isomorphic to $U(W_1)_{X_1}$, the stabilizer of X_1 under the action of $U(W_1)$. It is a product of the restriction of scalar of unitary group of lower dimension over an extension of F (including the general linear group). Let $\mathfrak{U}(W_1)_{X_1}$ be the respective Lie algebra. Then $U(W_1)_{X_1}$ acts on $\mathfrak{U}(W_1)_{X_1} \times W_1$. This representation of $U(W_1)_{X_1}$ is a product of representations of the same type (including the general linear case). The sliced representation at x is then isomorphic to the product of the above representation of $U(W_1)_{X_1}$ on $\mathfrak{U}(W_1)_{X_1} \times W_1$ and the representation of the trivial group on the normal space at (X_2, w) of $U(W_2)$ -orbit of (X_2, w) in $\mathfrak{U}(W_2) \times W_2$.

We have a similar description for \mathfrak{S}_n (cf. [51], [3]). Indeed we consider an equivalent version: the restriction of the adjoint action of $\text{GL}_{n+1, F}$ on $\mathfrak{gl}_{n+1, F}$ to $\text{GL}_{n, F}$. We describe the general form of a semisimple element in the Lie algebra $\mathfrak{gl}_{n+1, F}$. Let

$$(B.1) \quad x = \begin{pmatrix} X & u \\ v & d \end{pmatrix}, \quad X \in \mathfrak{gl}_{n, F},$$

where $u \in F^n$, $v \in F_n$ and we use F^n (F_n resp.) to denote the n -dimensional space of column (row, resp.) vectors. There is an obvious pairing between F^n and F_n . Let U_2 be the subspace of F^n spanned by $u, Xu, \dots, X^n u$ and similarly V_2 the subspace of F_n spanned by v, vX, \dots, vX^n . And let $V_1 := U_2^\perp \subset F^n$ ($U_1 := V_2^\perp$, resp.) be the orthogonal complement of U_2 (V_2 , resp.). Then for x to be semisimple, it is necessary that F_n (F^n , resp.) is the direct sum of U_2^\perp and V_2 (V_2^\perp and U_2 , resp.). Assuming this, according to the decomposition $F^n = U_1 \oplus U_2$, we may

write $X = \begin{pmatrix} X_{11} & X_{12} \\ 0 & X_{22} \end{pmatrix}$. Then by [51, §8], for such x to be semisimple, it is necessary and sufficient that $X_{11} \in \text{End}(U_1)$ is semisimple in the usual sense and $X_{12} = 0$.

Some auxiliary construction. Now we construct étale Luna slices for semisimple elements. As we shall see, the general case is basically a composition of two extreme cases:

- 1) the case $w = 0$ (namely, $r = 0$ minimal),
- 2) the case $x = (X, w)$ is regular semisimple (namely, $r = 0$ maximal).

The first case is essentially the same as the classical case (the Luna slice for the adjoint representation, cf. [40, §14]). For the second case we have to resort to the existence theorem of Luna. The general case can be reduced to those two basic cases.

We first describe a locally closed subvariety of $x = (X, w)$ following [51, §18]. The case for \mathfrak{S} is similar, following [51, §7]. Let (X, w) be as above. Then we have an orthogonal decomposition $W_1 \oplus W_2$. Denote $r = \dim W_2 \geq 1$. We define a closed subvariety Ξ of $\mathfrak{U}(W) \times W$ consisting of (Y, u) such that $u, Yu, \dots, Y^{r-1}u$ span W_2 . In particular, a semisimple $x = (X, w)$ belongs to Ξ . In [51, §18] Rallis and Schiffmann have defined an isomorphism of varieties

$$(B.2) \quad \iota_1 : \Xi \rightarrow (\mathfrak{U}(W_1) \times W_1) \times (\mathfrak{U}(W_2) \times W_2)_{rs}$$

whose inverse is defined as follows. According to the decomposition $W_1 \oplus W_2$ we may write Y as

$$\begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix}.$$

Then $Y_{12} \in \text{Hom}(W_2, W_1)$. Define $u' = Y_{12}Y^{r-1}u \in W_1$. Then ι_1^{-1} maps (Y, u) to $((Y_{11}, u'), (Y_{22}, u))$. It is then easy to see that $Y^i u = Y_{22}^i u \in W_2$ for $i = 0, 1, \dots, r-1$. Therefore (X_{22}, u) is a regular semisimple element. It is not hard to see that this defines an isomorphism. Moreover it is equivariant under the action of $U(W_1) \times U(W_2)$: on the left hand side the action is the restriction of that of $U(W)$ to the subgroup; on the right hand side the action is the product of the action of the two unitary groups $U(W_i)$ on $\mathfrak{U}(W_i) \times W_i$. Then the morphism ι induces a morphism between the categorical quotients

$$\iota_1^\sharp : (\mathfrak{U}(W_1) \times W_1) // U(W_1) \times (\mathfrak{U}(W_2) \times W_2)_{rs} // U(W_2) \rightarrow (\mathfrak{U}(W) \times W) // U(W).$$

Similarly, we have morphisms still denoted by ι_1 and ι_1^\sharp in the general linear case \mathfrak{S} . And we have an equivalent version for the restriction of the adjoint action of $\text{GL}_{n+1, F}$ on $\mathfrak{gl}_{n+1, F}$ to $\text{GL}_{n, F}$. We describe the analogous construction for this. Let $x \in \mathfrak{gl}_{n+1, F}$ be semisimple given by (B.1) and denote by $r = \dim U_2 = \dim V_2$. Without loss of generality, we may assume $U_2 = F^r$ embedded into F^n by sending u to $(0, \dots, 0, u)$. Similarly for V_2, U_1, V_1 . Then we define a closed subvariety Ξ consisting of

$$(B.3) \quad y = \begin{pmatrix} Y & u' \\ v' & d' \end{pmatrix}$$

such that $u', Y'u', \dots, Y^n u'$ span U_2 and $v', v'Y, \dots, v'Y^n$ span V_2 . Then similarly we have an isomorphism ([51, §7]):

$$(B.4) \quad \iota_1 : (\mathfrak{gl}_{n-r, F} \times F^{n-r} \times F_{n-r}) \times (\mathfrak{gl}_{r, F} \times F^r \times F_r)_{rs} \times F \rightarrow \Xi$$

such that ι^{-1} maps y to $((Y_{11}, Y_{12}Y_{22}^{r-1}u, vY_{22}^{r-1}Y_{21}), (Y_{22}, u, v), d)$. It also induces a morphism still denoted by

$$\iota_1^\sharp : (\mathfrak{gl}_{n-r} \times F^{n-r} \times F_{n-r}) // \mathrm{GL}_{n-r} \times (\mathfrak{gl}_r \times F^r \times F_r)_{rs} // \mathrm{GL}_r \times F \rightarrow \mathfrak{gl}_{n+1} // \mathrm{GL}_n.$$

Lemma B.1. *The morphism ι_1^\sharp is étale for both \mathfrak{gl}_{n+1} (equivalently, \mathfrak{S}_{n+1}) and $\mathfrak{U}(W) \times W$.*

Proof. We will use the coordinates described earlier for these categorical quotients involved in the morphism ι_1^\sharp . By Jacobian criterion for étaleness, it suffices to show that the Jacobian of ι_1^\sharp is non-zero everywhere. Indeed we will show the Jacobian is a non-zero constant. Therefore it suffices to compute the Jacobian over the algebraic closure. In particular, it suffices to consider the equivalent question for the $\mathrm{GL}_{n, F}$ -action on $\mathfrak{gl}_{n+1, F}$. Note that the Jacobian is a regular function on the source of the morphism ι_1^\sharp . It is then enough to show that it is a non-zero constant on a Zariski open subset. Recall that by Lemma 3.1, the categorical quotient $\mathfrak{gl}_{n+1, F} // \mathrm{GL}_{n, F}$ is given by

$$\mathrm{Spec}(F[\alpha_1, \dots, \alpha_{n+1}, \beta_1, \beta_2, \dots, \beta_n]),$$

where the invariants are defined by (3.4)

$$\alpha_i = \mathrm{tr} \wedge^i x, \quad \beta_j = \mathrm{ex}^j e^*, \quad x \in \mathfrak{gl}_{n+1, F}.$$

Another choice of invariants is given by (3.4) so that we may also identify the categorical quotient $\mathfrak{gl}_{n+1, F} // \mathrm{GL}_{n, F}$ as

$$\mathrm{Spec}(F[\alpha'_1, \dots, \alpha'_n, d, \beta'_1, \dots, \beta'_n]),$$

where

$$\alpha'_i = \mathrm{tr} \wedge^i X, \quad \beta'_j = vX^{j-1}u,$$

where x is as in (B.1). In particular, the Jacobian of the isomorphism

$$\psi = \psi_n : \mathrm{Spec}(F[\alpha'_1, \dots, \alpha'_n, d, \beta'_1, \dots, \beta'_n]) \rightarrow \mathrm{Spec}(F[\alpha_1, \dots, \alpha_{n+1}, \beta_1, \beta_2, \dots, \beta_n])$$

is a nonzero constant

$$(B.5) \quad \frac{\partial(\alpha_1, \dots, \alpha_{n+1}, \beta_1, \beta_2, \dots, \beta_n)}{\partial(\alpha'_1, \dots, \alpha'_n, d, \beta'_1, \dots, \beta'_n)} = \kappa_n \in F^\times.$$

To indicate the dependence on n we will write the invariants as $\alpha_i(n), \beta_j(n)$ etc..

We choose an auxiliary open subvariety of \mathfrak{gl}_{n+1} consisting of *strongly regular* elements in the sense of Jacquet–Rallis [39]. More precisely they are in the $\mathrm{GL}_{n,F}$ -orbits of elements of the form

$$x = x(a_1, \dots, a_{n+1}, c_1, \dots, c_n) := \begin{pmatrix} a_1 & 1 & & & & \\ c_1 & a_2 & 1 & & & \\ & \dots & \dots & & & \\ & & c_{n-1} & a_n & 1 & \\ & & & c_n & a_{n+1} & \end{pmatrix}, \quad a_i \in F, c_j \in F^\times.$$

All such x 's with $a_{n+1} = 0$ form a locally closed subvariety denoted by Θ_n of $\mathfrak{gl}_{n+1,F}$. Note that $\dim \Theta_n = 2n$. For $d \in F$, let $\delta_n(d)$ be the matrix with only non-zero entry d at the position $(n+1, n+1)$. For $(x, d) \in \Theta_n \times F$, taking invariants α_i, β_j of $x + \delta_n(d)$ yields a morphism with Zariski dense image:

$$\xi = \xi_n : \Theta_n \times F \rightarrow \mathrm{Spec}(F[\alpha_1, \dots, \alpha_{n+1}, \beta_1, \beta_2, \dots, \beta_n]).$$

Then we claim that the Jacobian of ξ_n is given by

$$(B.6) \quad \frac{\partial(\alpha_1, \dots, \alpha_n, d, \beta_1, \dots, \beta_n)}{\partial(a_1, \dots, a_n + 1, c_1, \dots, c_n)} = (-1)^{n(n-1)/2} \kappa_1 \kappa_2 \dots \kappa_n c_2 c_3^2 \dots c_n^{n-1}.$$

We prove (B.6) by induction on n . It is easy to verify this for $n = 1$. Now for $n > 1$, we may write $\xi_n = \xi'_n \circ \psi_n$. As $d = a_{n+1}$, we have

$$\frac{\partial(\alpha'_1, \dots, \alpha'_n, d, \beta'_1, \dots, \beta'_n)}{\partial(a_1, \dots, a_{n+1}, c_1, \dots, c_n)} = \frac{\partial(\alpha'_1, \dots, \alpha'_n, \beta'_1, \dots, \beta'_n)}{\partial(a_1, \dots, a_n, c_1, \dots, c_n)}.$$

Note that $\alpha'_i(n) = \alpha_i(n-1)$ and for $x \in \Theta_n$, $\beta'_1(n)(x) = c_n$ and

$$\beta'_j(n)(x) = c_n \beta_{j-1}(n-1)(X), \quad j \geq 2,$$

since $(0, \dots, 0, c_n) = c_n e_n$. Here X is as in (B.1). This gives us

$$\frac{\partial(\alpha'_1, \dots, \alpha'_n, \beta'_1, \dots, \beta'_n)}{\partial(a_1, \dots, a_n, c_1, \dots, c_n)} = \frac{\partial(\alpha'_1, \dots, \alpha'_n, \beta'_1, \dots, \beta'_{n-1})}{\partial(a_1, \dots, a_n, c_1, \dots, c_{n-1})}$$

which is equal to

$$(-1)^{n-1} c_n^{n-1} \frac{\partial(\alpha_1(n-1), \dots, \alpha_n(n-1), \beta_1(n-1), \dots, \beta_{n-1}(n-1))}{\partial(a_1, \dots, a_n, c_1, \dots, c_{n-1})}.$$

By induction hypothesis, the Jacobian of ξ'_n is:

$$(-1)^{n(n-1)/2} \kappa_1 \kappa_2 \dots \kappa_{n-1} c_2 c_3^2 \dots c_n^{n-1}.$$

Together with the Jacobian of ψ_n (B.5), we have proved (B.6).

Now we return to the morphism ι_1^\sharp . We have an obvious isomorphism

$$\phi : \Theta_{n-r} \times \Theta_r \times F \rightarrow \Theta_n \times F,$$

which sends the triple

$$(x(a_1, \dots, a_{n-r}, 0, c_1, \dots, c_{n-r}), x(a_{n-r+1}, \dots, a_n, 0, c_{n-r+1}, \dots, c_n), d)$$

to $x(a_1, \dots, a_{n+1}, c_1, \dots, c'_{n-r}, c_{n-r+1}, \dots, c_n)$ with

$$a_{n+1} = d, \quad c'_{n-r} = c_{n-r} \prod_{i=1}^r c_{n-r+i}.$$

We have the product

$$\xi_{n-r,r} : \Theta_{n-r} \times \Theta_r \times F \rightarrow (\mathfrak{gl}_{n-r} \times F^{n-r} \times F_{n-r}) // \mathrm{GL}_{n-r} \times (\mathfrak{gl}_r \times F^r \times F_r)_{rs} // \mathrm{GL}_r \times F.$$

It is to easy to see that the following diagram commutes:

$$\begin{array}{ccc} \Theta_{n-r} \times \Theta_r \times F & \xrightarrow{\phi} & \Theta_n \times F \\ \downarrow \xi_{n-r,r} & & \downarrow \xi_n \\ (\mathfrak{gl}_{n-r} \times F^{n-r} \times F_{n-r}) // \mathrm{GL}_{n-r} \times (\mathfrak{gl}_r \times F^r \times F_r)_{rs} // \mathrm{GL}_r \times F & \longrightarrow & \mathfrak{gl}_{n+1} // \mathrm{GL}_n \end{array}$$

Then the Jacobian of ι_1^\sharp restricted to the image of $\xi_{n-r,r}$ is equal to the ratio of the Jacobian of ξ_n over the product of that of $\xi_{n-r,r}$ and ϕ . By (B.6), we obtain this ratio is a non-zero constant times

$$\frac{c_2 c_3^3 \dots c_n^{n-1}}{c_2 c_3^2 \dots (c_{n-r} \prod_{i=1}^r c_{n-r+i})^{n-r-1} \cdot c_{n-r+2} c_{n-r+3}^2 \dots c_n^{r-1} \cdot \prod_{i=1}^r c_{n-r+i}} = 1.$$

This shows that the Jacobian of ι_1^\sharp is a non-zero constant on a Zariski dense subset and hence itself a non-zero constant on the source of ι_1^\sharp . This completes the proof that ι_1^\sharp is étale. \square

The construction of étale Luna slices. We now refine the morphism ι_1 defined by (B.2) and (B.4). We consider the semisimple element $x = (X, w)$ in the unitary case. Then X_{11} is semisimple in the usual sense. Therefore we may consider the Lie algebra $\mathfrak{u}(W_1)_{X_{11}}$ of the stabilizer $U(W_1)_{X_{11}}$ and an open subvariety $\mathfrak{u}(W_1)'_{X_{11}}$ of $\mathfrak{u}(W_1)_{X_{11}}$ consisting of those Y such that (cf. [40, 14.5])

$$(B.7) \quad \det(\mathrm{ad}(Y); \mathfrak{u}(W_1) / \mathfrak{u}(W_1)_{X_{11}}) \neq 0.$$

To simplify notations, for $\mathfrak{u}(V)$ we denote

$$(B.8) \quad \mathcal{V} = \mathfrak{u}(W) \times W, \quad \mathcal{V}_x = (\mathfrak{u}(W_1)_{X_{11}} \times W_1) \times (\mathfrak{u}(W_2) \times W_2),$$

and

$$(B.9) \quad H = U(W), \quad H_i = U(W_i), \quad H_x = U(W_1)_{X_{11}},$$

where the last group is isomorphic to the stabilizer of x . Set

$$(B.10) \quad \mathcal{V}'_x = (\mathfrak{U}(W_1)'_{X_{11}} \times W_1) \times (\mathfrak{U}(W_2) \times W_2)_{rs}.$$

Then $x \in \mathcal{V}'_x$. Let ι_2 be the restriction of ι_1 to \mathcal{V}'_x . Then the morphism ι_2 is H_x -equivariant and it induces a morphism

$$(B.11) \quad \iota_2^\sharp : \mathcal{V}'_x // H_x \times H_2 \rightarrow \mathcal{V} // H.$$

Note that the morphism ι_2 also induces a morphism

$$(B.12) \quad \iota : H \times_{(H_x \times H_2)} \mathcal{V}'_x \rightarrow \mathcal{V}$$

by sending (h, x) to $h \cdot \iota_2(x)$. Similar construction applies to \mathfrak{gl}_{n+1} (equivalently, \mathfrak{S}).

Lemma B.2. *The morphism ι_2^\sharp is étale for both \mathfrak{gl}_{n+1} (equivalently, \mathfrak{S}) and $\mathfrak{U}(V)$.*

Proof. By Lemma, it suffices to show that the morphism

$$(\mathfrak{U}(W_1)'_{X_{11}} \times W_1) // U(W_1)_{X_{11}} \rightarrow (\mathfrak{U}(W_1) \times W_1) // U(W_1)$$

is étale. It is not hard to show that the Jacobian of this morphism at the image of $(Y, u) \in \mathfrak{U}(W_1)'_{X_{11}} \times W_1$ is given by, up to a sign:

$$\det(\text{ad}(Y); \mathfrak{U}(W_1) / \mathfrak{U}(W_1)_{X_{11}}).$$

This is non-zero by the definition of $\mathfrak{U}(W_1)'_{X_{11}}$. □

Lemma B.3. *The morphism ι is étale.*

Proof. We show this in the unitary case. It suffices to show that the differential $d\iota$ at $(1, y)$ induces an isomorphism between the tangent spaces. We first assume that X_{11} is a scalar. Then $H_x = H_1$. Suppose $y = (Y, u) \in \mathcal{V}_x$. Then it is not hard to see that we have for $\Delta Y = \text{diag}(\Delta Y_{11}, \Delta Y_{22})$, $\Delta u = (\Delta u_1, \Delta u_2)$,

$$\iota_2(Y + \Delta Y, u + \Delta u) = \iota(Y, u) + \Delta \iota_2(Y, u) + \text{higher terms},$$

where

$$\Delta \iota_2(Y, u) := \begin{pmatrix} \Delta Y_{11} & \phi_{\Delta u_1} \\ \dots & \Delta Y_{22} \end{pmatrix},$$

where the part “ \dots ” is determined by the Hermitian condition, and $\phi_{\Delta u_1} \in \text{Hom}(W_2, W_1)$ is the homomorphism that sends $Y_{22}^i u_2$ to 0 for $i = 0, 1, \dots, r-2$ and $Y_{22}^{r-1} u_2$ to Δu_1 .

Then the differential $d\iota$ at $(1, y)$ is given by

$$\begin{aligned} d\iota : \mathfrak{U}(W) \times_{\mathfrak{U}(W_1)_{X_{11}} \times \mathfrak{U}(W_2)} \mathcal{V}_x &\rightarrow \mathcal{V} \\ (\Delta X, (\Delta Y, \Delta u)) &\mapsto ([\Delta X, \text{diag}(Y_{11}, Y_{22})] + \Delta \iota_2(Y, u), \Delta u_2 + \Delta X \cdot u_2). \end{aligned}$$

Here the left hand side means the quotient of $(\mathfrak{U}(W) \times \mathcal{V}_x)/\mathfrak{U}(W_1)_{X_{11}} \times \mathfrak{U}(W_2)$. By comparing the dimension, it suffices to show that $d\iota$ is injective. Suppose that $d\iota(\Delta X, (\Delta Y, \Delta u)) = 0$. By the action of $(\mathfrak{U}(W) \times \mathcal{V}_x)/\mathfrak{U}(W_1)_{X_{11}} \times \mathfrak{U}(W_2)$, we may assume that

$$\Delta X = \begin{pmatrix} 0 & \phi \\ \dots & 0 \end{pmatrix}, \phi \in \text{Hom}_E(W_2, W_1).$$

Then we need to show that $\phi = 0$ and $(\Delta Y, \Delta u) = 0$. From the diagonal blocks, it is easy to see that $\Delta Y = 0$. Note that now we have $\Delta X \cdot u_2 = \phi(u_2) \in W_1$. Therefore $\Delta u_2 + \Delta X \cdot u_2 = 0$ implies that both $\Delta u_2 = 0$ and $\phi(u_2) = 0$. Now we use the condition from the off-diagonal block to obtain

$$Y_{11}\phi - \phi Y_{22} + \phi_{\Delta u_1} = 0 \in \text{Hom}_E(W_2, W_1).$$

Since (Y_{22}, u_2) is regular semisimple, the vectors $u_2, Y_{22}u_2, \dots, Y_{22}^{r-1}u_2$ form a basis of W_2 . We apply the above homomorphism to $Y_{22}^i u_2$:

$$\phi Y_{22}^{i+1} u_2 = Y_{11} \phi Y_{22}^i u_2 + \phi_{\Delta u_1} Y_{22}^i u_2.$$

Since $\phi u_2 = 0$, we may show that $\phi Y_{22}^{i+1} u_2 = 0$ recursively. This shows that $\phi = 0 \in \text{Hom}(W_2, W_1)$. This completes the proof when X_{11} is a scalar.

Now we consider a general semisimple X_{11} , and $H_x \simeq U(W_1)_{X_{11}}$. Then the assertion follows if we show that the following analogous morphism is étale

$$H_1 \times_{H_x} (\mathfrak{U}(W_1)'_{X_{11}} \times W_1) \rightarrow \mathfrak{U}(W_1) \times W_1.$$

Similar argument to the above works and we omit the details. \square

Lemma B.4. *For both $\mathfrak{gl}_{n+1, F}$ (equivalently, \mathfrak{S}) and $\mathfrak{U}(V)$, the following diagram is cartesian*

$$\begin{array}{ccc} H \times_{(H_x \times H_2)} \mathcal{V}'_x & \xrightarrow{\iota} & \mathcal{V} \\ \downarrow & & \downarrow \pi \\ \mathcal{V}'_x // (H_x \times H_2) & \xrightarrow{\iota_2^\sharp} & \mathcal{V} // H. \end{array}$$

Proof. Thanks to the previous lemmas, now the proof is similar to that of [40, Lemma 14.1]. We need to show that the induced morphism

$$\gamma : H \times_{(H_x \times H_2)} \mathcal{V}'_x \rightarrow \mathcal{V}'_x // (H_x \times H_2) \times_{\mathcal{V}/H} \mathcal{V}$$

is an isomorphism. It suffices to show that in an algebraic closure the induced map on the geometric points is bijective. From this we see that the question becomes equivalent for both \mathfrak{gl}_{n+1} (equivalently, \mathfrak{S}) and $\mathfrak{U}(W)$. To simplify exposition, we consider the unitary case.

Actually the bijectivity holds for any field as we now show. To show the surjectivity, we may write an element $\mathcal{V}'_x // (H_x \times H_2) \times_{\mathcal{V}/H} \mathcal{V}$ as $((Y', u'), (Y, u))$ where $(Y', u') \in \mathcal{V}'_x$ is abused to denote its image in the quotient. Then $\iota_2^\sharp(Y', u') = \pi(Y, u)$. This implies that the fundamental matrix of $u, Yu, \dots, Y^{r-1}u$ is equal to that of $u'_2, Y'_{22}u'_2, \dots, Y'^{r-1}_2 u'_2$, which is non-degenerate.

By Witt's theorem there exists an $h \in H$ such that $h(Y_{22}^i u'_2) = Y^i u$ for $0 \leq i \leq r-1$. We may thus assume that $u = u'_2, Y_{22} = Y'_{22}$. The rest follows from [40, Lemma 14.1].

To show the injectivity, without loss of generality it suffices to show that if

$$\gamma(1, y) = \gamma(h, z),$$

then $h \in H_x \times H_2$ and $y = hz$. It suffices to show $h \in H_x \times H_2$ since the second assertion follows from this. Denote $y = (Y, u)$ and $z = (Z, w)$. Then h obviously preserves the subspace W_2 and hence W_1 , too. It follows that $h \in H_1 \times H_2$. The rest follows from [40, Lemma 14.1]. \square

Theorem B.5. *Let $x \in \mathcal{V}$ be a semisimple element and we use the notations from (B.8) to (B.10). Choose an étale Luna slice Z_2 of (X_{22}, w) (which is H_2 -regular semisimple) for the action of H_2 on $\mathfrak{U}(W_2) \times W_2$. Then the image of $(\mathfrak{U}(W_1)'_{X_{11}} \times W_1) \times Z_2$ under ι_2 defines an étale Luna slice at x .*

Proof. The space \mathcal{V}_x in (B.8) is (isomorphic to) the sliced representation at x . Then the result follows from Lemma B.3 and Lemma B.4. \square

Remark 17. One may make the étale slice for a regular semisimple element more explicit by using the explicit section of the categorical quotient in Lemma 3.1.

Remark 18. Obviously, this also gives us a way to choose an analytic Luna slice once we choose an analytic Luna slice for the H_2 -regular semisimple element (X_{22}, w) .

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